THE UNIVERSITY OF ALBERTA

THE APPROXIMATE DISTRIBUTION OF THE CORRELATION BETWEEN TWO STATIONARY LINEAR MARKOV SERIES WITH FITTED MEANS

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ABSTRACT

The purpose of this thesis is to obtain the approximate null distribution of the sample product-moment correlation r for the two stationary, linear Markov series $\{x_i\}$ and $\{y_i\}$, where the means are fitted and the auto-correlations are ρ_1 and ρ_2, respectively. The relevant literature is reviewed in Chapter I. Chapter II contains a detailed discussion of the derivation of the approximate null distribution of r where the means of the two stationary linear Markov series are known. In Chapter III the actual approximations of the density function of r for the case where the means are fitted is obtained and illustrated graphically.

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CHAPTER 1

INTRODUCTION

Let (x_1,y_1), (x_2,y_2), ..., (x_n,y_n) be a sample from a bivariate normal population with correlation ρ. If either $(x_1,x_2, ..., x_n)$ or $(y_1,y_2, ..., y_n)$ is a series of independent observations, the sample product-moment correlation r has the familiar null distribution $(\rho=0)$, (see Keeping [7]), but, if both series are serially correlated, very little is known about the distribution of r. Orcutt and James [12] have made the following remarks relevant to this problem:

> The testing of the significance of a correlation involves a comparison with what would have been obtained between non-related series thought to be analogous to the observed series. .....
>
> There is, however, one obvious point at which the analogy underlying such tests of significance may break down when one is concerned with economic time series, namely, if the consecutive terms are really correlated. In economic time series, in metereological time or spatial series, or, for that matter, in biological time series, auto-correlation usually exists. .....
>
> In economics, most of the material that we wish to investigate for relationships exhibits auto-correlation, and there is a real need for a test of significance for correlations which is based on a more realistic sampling model.

The literature pertaining to this subject is sparse. For a concise review of some of the earlier studies of the distribution of r, we quote McGregor's paper [9]:

> When both series are of the stationary Markov type with auto-correlations ρ_1 and ρ_2, Bartlett (1935), [2],
>
> has shown that the approximate variance of r is given by

$$\mathrm{var}(r) \sim \frac{1}{n} \left[\frac{1 + \rho_1\rho_2}{1 - \rho_1\rho_2}\right] .$$

Orcutt and James (1948), [12], concluded, from the results of a sampling experiment, that there is some evidence that the variance of the correlation between two time-series is nearly independent of the true auto-correlations. In another sampling experiment, Quenouille (1949), [15], worked out various partial cross-correlations between two time-series with $\rho_1 = 0.6$ and $\rho_2 = 0.4$. In particular, the observed variance of $r(x_2y_2|x_1y_1)$, the partial cross-correlation between x_j and y_j with the effects x_{j-1} and y_{j-1} removed, was considerably less than that of r. Hannan (1955), [6], has shown that $r(x_2y_2|x_1y_1)$ provides a more efficient test of $\rho = 0$ than does r.

Using an approach due to Daniels [3], [4] based on the method of steepest descents, McGregor [9] obtained the approximate null distribution of r, when the series $\{x_i\}$ and $\{y_i\}$ are each of the stationary linear Markov type with known means and auto-correlations ρ_1 and ρ_2, respectively.

In this thesis we shall consider the more general case where the means are fitted, and derive the approximate null distribution of r for the two stationary, linear Markov processes $\{x_i\}$ and $\{y_i\}$ with serial correlations ρ_1 and ρ_2, respectively.

As conjectured by McGregor, the analysis of [9] can be extended to the fitted means case. In Chapter II we discuss McGregor's method for the known means case in detail in order to keep clarity and unity throughout the development of the more general case where the means are fitted. The approximate density function $(p^*(r))$ of the product-moment correlation r for the fitted means case (as for the known means case [9]) was [illegible] to depend on the auto-correlations ρ_1 and ρ_2 only through their product,

[see (3.33) (also (2.61))]. Upon renormalization of $p^*(r)$, (3.46), we observe that it is of the same form as $p(r)$, the approximate density function of r for the known means case, with N replaced by $(M-1)$. The main problem in this development was the approximation of the determinant $|\underline{B}|$ encountered in the joint-moment generating function of the quadratic forms used in defining r, (see Appendix III B).

CHAPTER II

THE APPROXIMATE DISTRIBUTION OF THE CORRELATION BETWEEN TWO STATIONARY, LINEAR MARKOV SERIES WITH KNOWN MEANS

In this chapter we discuss in detail McGregor's [9] derivation of the approximate null distribution of r when the series $\{x_i\}$ and $\{y_i\}$ are each of the stationary, linear Markov type with known means and with autocorrelations ρ_1 and ρ_2, respectively.

Consider the two stationary, linear Markov processes

$$x_i = \rho_1 x_{i-1} + \epsilon_i \qquad \text{for all } i$$

and

$$y_j = \rho_2 y_{j-1} + \eta_j \qquad \text{for all } j ,$$

where the ϵ's and η's are independent $N(0,1)$ variables. The joint distribution of $x_1, x_2, \ldots, x_n$ is

$$dF_1(x_1,\ldots,x_n) = (2\pi)^{-\frac{n}{2}} \left|\Sigma_x\right|^{-\frac{1}{2}} \exp\left[-\tfrac{1}{2}\, \underline{x}' \left|\Sigma_x\right|^{-1} \underline{x}\right] dx_1 \ldots dx_n ,$$

where $\underline{x} = (x_1,x_2,\ldots,x_n)'$ and $\Sigma_x^{(n\times n)}$ is the covariance matrix of $\underline{x}$. For the stationary, linear Markov process it is easily shown that

$$\operatorname{var}(x_i) = \frac{1}{1-\rho_1^2}$$

and

$$\operatorname{cov}(x_{i-k},x_i) = \frac{\rho_1^{|k|}}{1-\rho_1^2} \qquad \text{(for any fixed } k) .$$

Hence, the covariance matrix for $\underline{x}$ is

$$\Sigma_x = \begin{bmatrix} \frac{1}{1-\rho_1^2} & \frac{\rho_1}{1-\rho_1^2} & \frac{\rho_1^2}{1-\rho_1^2} & \cdots & \frac{\rho_1^{n-1}}{1-\rho_1^2} \\ \frac{\rho_1}{1-\rho_1^2} & \frac{1}{1-\rho_1^2} & \frac{\rho_1}{1-\rho_1^2} & \cdots & \frac{\rho_1^{n-2}}{1-\rho_1^2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\rho_1^{n-1}}{1-\rho_1^2} & \frac{\rho_1^{n-2}}{1-\rho_1^2} & \frac{\rho_1^{n-3}}{1-\rho_1^2} & \cdots & \frac{1}{1-\rho_1^2} \end{bmatrix} . \tag{2.1}$$

Also, as in Patton's thesis [13], it can be shown that

$$|\Sigma_x| = \frac{1}{1-\rho_1^2} \tag{2.2}$$

and

$$\Sigma_x^{-1} = \begin{bmatrix} 1 & -\rho_1 & 0 & \cdots & 0 & 0 \\ -\rho_1 & 1+\rho_1^2 & -\rho_1 & \cdots & 0 & 0 \\ 0 & -\rho_1 & 1+\rho_1^2 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1+\rho_1^2 & -\rho_1 \\ 0 & 0 & 0 & \cdots & -\rho_1 & 1 \end{bmatrix} , \tag{2.3}$$

and hence,

$$\underline{x}' \, \Sigma_x^{-1} \, \underline{x} = x_1^2 - 2\rho_1(x_1x_2 + x_2x_3 + \cdots + x_{n-1}x_n)$$

$$+ (1+\rho_1^2)(x_2^2 + \cdots + x_{n-1}^2) + x_n^2 \; .$$

Similarly, the joint distribution of $y_1, y_2, \ldots, y_n$ is

$$dF_2(y_1,\ldots,y_n) = (2\pi)^{-\frac{n}{2}} \, |\Sigma_y|^{-\frac{1}{2}} \exp\left[-\tfrac{1}{2}\, \underline{y}' \, \Sigma_y^{-1} \, \underline{y}\right] dy_1 \ldots dy_n \; ,$$

where $\underline{y} = (y_1, y_2, \ldots, y_n)'$ and $\Sigma_y^{(n\times n)}$ is the covariance matrix of $\underline{y}$ which is of the same form as (2.1) except that ρ_1 is replaced by ρ_2. Also, $|\Sigma_y|$ and Σ_y^{-1} are given by (2.2) and (2.3), respectively, with ρ_1 replaced by ρ_2. Hence,

$$\begin{aligned} \underline{y}' \, \Sigma_y^{-1} \, \underline{y} = y_1^2 &- 2\rho_2(y_1 y_2 + y_2 y_3 + \ldots + y_{n-1} \, y_n) \\ &+ (1 + \rho_2^2)(y_2^2 + \ldots + y_{n-1}^2) + y_n^2 \ . \end{aligned}$$

Now the joint distribution of $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n$ is

$$\begin{aligned} dF(x_1, \ldots, x_n, y_1, \ldots, y_n) &= (2\pi)^{-\frac{n}{2}} |\Sigma_x|^{-\frac{1}{2}} \quad (2\pi)^{-\frac{n}{2}} \quad |\Sigma_y|^{-\frac{1}{2}} \\ &\times \exp\left\{-\tfrac{1}{2}\left[\underline{x}' \ \Sigma^{-1} \ \underline{x} + \underline{y}' \ \Sigma_y^{-1} \ \underline{y}\right]\right\} \\ &\times dx_1 \ldots dx_n dy_1 \ldots dy_n \\ &= \frac{(1-\rho_1^2)^{\frac{1}{2}}(1-\rho_2^2)^{\frac{1}{2}}}{(2\pi)^n} \\ &\times \exp\left\{-\tfrac{1}{2}\left[x_1^2 - 2\rho_1(x_1 x_2 + x_2 x_3 + \ldots + x_{n-1} x_n)\right.\right. \\ &\qquad + (1+\rho_1^2)(x_2^2 + \ldots + x_{n-1}^2) + x_n^2 + y_1^2 \\ &\qquad - 2\rho_2(y_1 y_2 + y_2 y_3 + \ldots + y_{n-1} y_n) \\ &\qquad \left.\left. + (1+\rho_2^2)(y_2^2 + \ldots + y_{n-1}^2) + y_n^2\right]\right\} \\ &\times dx_1 \ldots dx_n dy_1 \ldots dy_n \ . \qquad (2.4) \end{aligned}$$

We seek an approximate distribution of the product-moment correlation

$$r = \sqrt{r_1 \, r_2} \quad ,$$

where

$$r_1 = \frac{\sum_{i=1}^{n} x_i\, y_i}{\sum_{i=1}^{n} x_i^2} \quad \text{and} \quad r_2 = \frac{\sum_{i=1}^{n} x_i\, y_i}{\sum_{i=1}^{n} y_i^2} \ .$$

Let

$$C = \sum_{i=1}^{n} x_i^2 \ , \quad D = \sum_{i=1}^{n} y_i^2 \quad \text{and} \quad E = \sum_{i=1}^{n} x_i\, y_i \ .$$

The joint moment-generating function for the distribution of C, D and E is

$$M(T,S,U) = \mathcal{E}\,[e^{TC+SD+UE}]$$

$$= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{TC+SD+UE}\, dF$$

$$= \frac{(1-\rho_1^2)(1-\rho_2^2)}{(2\pi)^n} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp\left\{TC+SD+UE-\tfrac{1}{2}\left[\underline{x}'\, \Sigma_x^{-1}\, \underline{x} + \underline{y}'\, \Sigma_y^{-1}\, \underline{y}\right]\right\}$$

$$\times\, dx_1 \ldots dx_n dy_1 \ldots dy_n \ . \quad (2.5)$$

The exponent of the integrand is

$$TC+SD+UE - \tfrac{1}{2}\left[\underline{x}'\, \Sigma_x^{-1}\, \underline{x} + \underline{y}'\, \Sigma_y^{-1}\, \underline{y}\right]$$

$$= -\tfrac{1}{2}\left[\underline{x}'\, \Sigma_x^{-1}\, \underline{x} + \underline{y}'\, \Sigma_y^{-1}\, \underline{y} - 2TC - 2SD - 2UE\right]$$

$$= -\tfrac{1}{2}\Big[x_1^2 + (1+\rho_1^2)(x_2^2 + \ldots + x_{n-1}^2) + x_n^2 + y_1^2 + (1+\rho_2^2)(y_2^2+\ldots+y_{n-1}^2) + y_n^2$$

$$- 2\rho_1(x_1x_2+\ldots+x_{n-1}x_n) - 2\rho_2(y_1y_2+\ldots+y_{n-1}y_n)$$

$$- 2T(x_1^2+\ldots+x_n^2) - 2S(y_1^2+\ldots+y_n^2) - 2U(x_1y_1+\ldots+x_ny_n)\Big]$$

$$= -\tfrac{1}{2}\left(\frac{\underline{x}}{\underline{y}}\right)' \underline{A} \left(\frac{\underline{x}}{\underline{y}}\right)$$

$$= -\tfrac{1}{2}\, \underline{x}^{*'}\, \underline{A}\, \underline{x}^{*} \ ,$$

where $\underline{x}^* = \left(\frac{\underline{x}}{\underline{y}}\right) = (x_1,\ldots,x_n,y_1,\ldots,y_n)'$ and $\underline{A}$ is the $2n \times 2n$ partitioned matrix,

$$\underline{A} = \left[\begin{array}{c|c} \underline{Q}_1 & \underline{Q}_3 \\ \hline \underline{Q}_3 & \underline{Q}_2 \end{array}\right] ,$$

with $n \times n$ submatrices,

$$\underline{Q}_1 = \begin{bmatrix} 1-2T & -\rho_1 & 0 & \ldots & 0 & 0 \\ -\rho_1 & 1+\rho_1^2-2T & -\rho_1 & \ldots & 0 & 0 \\ 0 & -\rho_1 & 1+\rho_1^2-2T & \ldots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \ldots & 1+\rho_1^2-2T & -\rho_1 \\ 0 & 0 & 0 & \ldots & -\rho_1 & 1-2T \end{bmatrix} ,$$

$$\underline{Q}_2 = \begin{bmatrix} 1-2S & -\rho_2 & 0 & \ldots & 0 & 0 \\ -\rho_2 & 1+\rho_2^2-2S & -\rho_2 & \ldots & 0 & 0 \\ 0 & -\rho_2 & 1+\rho_2^2-2S & \ldots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \ldots & 1+\rho_2^2-2S & -\rho_2 \\ 0 & 0 & 0 & \ldots & -\rho_2 & 1-2S \end{bmatrix}$$

and $\underline{Q}_3$ the scalar matrix with diagonal elements $(-U)$.

Thus (2.5) is

$$M(T,S,U) = \frac{(1-\rho_1^2)^{\frac{1}{2}}(1-\rho_2^2)^{\frac{1}{2}}}{(2\pi)^n} \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \exp[-\tfrac{1}{2}\, \underline{x}^{*\prime}\, \underline{A}\, \underline{x}^*]$$

$$\times\, dx_1\ldots dx_n dy_1\ldots dy_n \,.$$

Since $\underline{A}$ is a symmetric, nonsingular $2n \times 2n$ matrix and $\underline{x}^{*\prime} = (x_1,\ldots,x_n,$ $y_1,\ldots,y_n)$, there exists an orthogonal transformation of the variables

$x_1,\ldots,x_n,y_1,\ldots,y_n$ which diagonalizes $\underline{A}$. Making this transformation and performing the integration, we get

$$M(T,S,U) = \frac{(1-\rho_1^2)^{\frac{1}{2}}(1-\rho_2^2)^{\frac{1}{2}}}{|\underline{A}|^{\frac{1}{2}}} , \tag{2.6}$$

where $|\underline{A}|^{-\frac{1}{2}}$ is the Jacobian of the transformation.

In Appendix III, (III.23), the determinant of $\underline{A}$ is evaluated approximately giving

$$|\underline{A}| \sim \rho_1^n \rho_2^n \frac{[a_2(\beta_{11}-a_2) - a_2^2(\beta_{12}-a_1)(\beta_{21}-a_1)]^2}{a_2^n(1-a_1+a_2)(1+a_1+a_2)(1-a_2)^2} , \tag{2.7}$$

where terms which will ultimately become exponentially small have been omitted, and

$$\left.\begin{aligned} \frac{1+a_1^2+a_2^2}{a_2} &= 2 + \frac{(1+\rho_1^2-2T)(1+\rho_2^2-2S)}{\rho_1\rho_2} - \frac{U^2}{\rho_1\rho_2} , \\ \frac{a_1(1+a_2)}{a_2} &= - \frac{(1+\rho_1^2-2T)}{\rho_1} - \frac{(1+\rho_2^2-2S)}{\rho_2} , \end{aligned}\right\} \tag{2.8}$$

$$\left.\begin{aligned} \beta_{11} &= \frac{(1-2T)(1-2S)}{\rho_1\rho_2} + 1 - \frac{U^2}{\rho_1\rho_2} , \\ \beta_{12} &= - \frac{(1-2T)}{\rho_1} - \frac{(1+\rho_2^2-2S)}{\rho_2} \end{aligned}\right.$$

and

$$\left.\begin{aligned} \beta_{21} &= - \frac{(1+\rho_1^2-2T)}{\rho_1} - \frac{(1-2S)}{\rho_2} . \end{aligned}\right\} \tag{2.9}$$

From (2.6) and (2.7) we obtain

$$M(T,S,U) \sim \frac{(1-\rho_1^2)^{\frac{1}{2}}(1-\rho_2^2)^{\frac{1}{2}} \quad a_2^{\frac{n}{2}}}{(\rho_1\rho_2)^{\frac{n}{2}}} \cdot \frac{(1-a_1+a_2)^{\frac{1}{2}}(1+a_1+a_2)^{\frac{1}{2}}(1-a_2)}{[a_2(\beta_{11}-a_2)-a_2^2(\beta_{12}-a_1)(\beta_{21}-a_1)]} . \tag{2.10}$$

Following the development of Appendix I, we make the substitution

$$U = \frac{v(r_1+r_2)}{r_1 r_2} - \frac{T}{r_1} - \frac{S}{r_2} \quad , \tag{2.11}$$

where $r_1 = \frac{E}{C}$ and $r_2 = \frac{E}{D}$. By the relations (2.8) and (2.11) a_1 and a_2 are implicit functions of T, S and v $(a_1 \equiv a_1(T,S,v)$, $a_2 \equiv a_2(T,S,v))$ and hence

$$\frac{\partial^2}{\partial v^2}\left[M\left(T,S,\ \frac{v(r_1+r_2)-Tr_2-Sr_1}{r_1 r_2}\right)\right]$$

$$\sim \frac{(1-\rho_1^2)^{\frac{1}{2}}(1-\rho_2^2)^{\frac{1}{2}}\frac{n}{2}(\frac{n}{2}-1)a_2^{\frac{n}{2}-2}}{(\rho_1\rho_2)^{\frac{n}{2}}} \cdot \frac{(1-a_1+a_2)^{\frac{1}{2}}(1+a_1+a_2)^{\frac{1}{2}}(1-a_2)}{[a_2(\beta_{11}-a_2)-a_2^2(\beta_{12}-a_1)(\beta_{21}-a_1)]}\left|\frac{\partial a_2}{\partial v}\right|^2 ,$$

where the error is relatively $O(\frac{1}{n})$.

Then using the inversion formula (I.3), we obtain the following joint density of r_1 and r_2:

$$h(r_1,r_2) \sim \frac{(1-\rho_1^2)^{\frac{1}{2}}(1-\rho_2^2)^{\frac{1}{2}}\ \frac{n}{2}(\frac{n}{2}-1)}{(\rho_1\rho_2)^{\frac{n}{2}}\ \ (r_1+r_2)^2}$$

$$\times \frac{1}{(2\pi i)^2}\iint a_2^{\frac{n}{2}-2}\ \frac{(1-a_1+a_2)^{\frac{1}{2}}(1+a_1+a_2)^{\frac{1}{2}}(1-a_2)}{[a_2(\beta_{11}-a_2)-a_2^2(\beta_{12}-a_1)(\beta_{21}-a_1)]}\left|\frac{\partial a_2}{\partial v}\right|^2\Bigg|_{v=0} dS\,dT, \tag{2.12}$$

where terms which are relatively $O(\frac{1}{n})$ have been ignored.

To evaluate the integral,

$$\frac{1}{(2\pi i)^2}\iint a_2^{\frac{n}{2}-2}\frac{(1-a_1+a_2)^{\frac{1}{2}}(1+a_1+a_2)^{\frac{1}{2}}(1-a_2)}{[a_2(\beta_{11}-a_2)-a_2^2(\beta_{12}-a_1)(\beta_{21}-a_1)]}\left(\frac{\partial a_2}{\partial v}\right)^2\Bigg|_{v=0} dSdT, \quad (2.13)$$

we use the following bivariate, saddle-point approximation:

$$\frac{1}{(2\pi i)^2}\iint [\psi(z_1,z_2)]^k \; \varphi(z_1,z_2)\; dz_2\; dz_1$$

$$\sim \frac{\varphi(\hat{z}_1,\hat{z}_2)\;[\psi(\hat{z}_1,\hat{z}_2)]^{k+1}}{2\pi k\left\{\psi_{11}(\hat{z}_1,\hat{z}_2)\psi_{22}(\hat{z}_1,\hat{z}_2)-[\psi_{12}(\hat{z}_1,\hat{z}_2)]^2\right\}^{\frac{1}{2}}}, \quad (2.14)$$

where
$$\psi_{11}(\hat{z}_1,\hat{z}_2) = \frac{\partial^2\,\psi(z_1,z_2)}{\partial z_1^2}\Bigg|_{z_1=\hat{z}_1,\,z_2=\hat{z}_2},$$

$$\psi_{22}(\hat{z}_1,\hat{z}_2) = \frac{\partial^2\,\psi(z_1,z_2)}{\partial z_2^2}\Bigg|_{z_1=\hat{z}_1,\,z_2=\hat{z}_2},$$

$$\psi_{12}(\hat{z}_1,\hat{z}_2) = \frac{\partial^2\,\psi(z_1,z_2)}{\partial z_1\,\partial z_2}\Bigg|_{z_1=\hat{z}_1,\,z_2=\hat{z}_2}$$

and where $\hat{z}_1$ and $\hat{z}_2$ are solutions of $\dfrac{\partial\psi(z_1,z_2)}{\partial z_1} = 0$ and $\dfrac{\partial\psi(z_1,z_2)}{\partial z_2} = 0$.

In (2.13) T and S correspond to z_1 and z_2 in (2.14), respectively, $a_2|_{v=0}$ to $\psi(z_1,z_2)$, $(\frac{n}{2}-2)$ to k and

$$\left\{\frac{(1-a_1+a_2)^{\frac{1}{2}}(1+a_1+a_2)^{\frac{1}{2}}(1-a_2)}{[a_2(\beta_{11}-a_2)-a_2^2(\beta_{12}-a_1)(\beta_{21}-a_1)]}\left(\frac{\partial a_2}{\partial v}\right)^2\Bigg|_{v=0}\right\} \text{ to } \varphi(z_1,z_2).$$

The paths of integration in the T and S planes are taken as the lines of steepest descent of $a_2|_{v=0} \equiv a_2(T,S)$ such that

$$\frac{\partial a_2(\hat{T},\hat{S})}{\partial \hat{T}} = \frac{\partial a_2(\hat{T},\hat{S})}{\partial \hat{S}} = 0 \ .$$

(Note that there is a distinction between $a_2 \equiv a_2(T,S,v)$ and $a_2|_{v=0} \equiv a_2(T,S)$, similarly between $a_1 \equiv a_1(T,S,v)$ and $a_1|_{v=0} \equiv a_1(T,S)$, but throughout the discussion it will be apparent which of these functions is used.)

Now putting $v = 0$, so that a_2 and a_1 are functions of T and S only, and eliminating a_1 from equations (2.8), we may write

$$\frac{1+a_2^2}{a_2} + \frac{a_2}{(1+a_2)^2}\left[\frac{(1+\rho_1^2-2T)}{\rho_1} + \frac{(1+\rho_2^2-2S)}{\rho_2}\right]^2$$

$$= 2 + \frac{(1+\rho_1^2-2T)(1+\rho_2^2-2S)}{\rho_1\rho_2} - \frac{1}{\rho_1\rho_2}\left[\frac{T}{r_1} + \frac{S}{r_2}\right]^2 . \qquad (2.14)$$

Differentiating (2.14) implicitly with respect to T(holding S constant) and setting $\frac{\partial a_2}{\partial T} = 0$, we obtain

$$\frac{-4\hat{a}_2}{(1+\hat{a}_2)^2\rho_1}\left[\frac{(1+\rho_1^2-2\hat{T})}{\rho_1} + \frac{(1+\rho_2^2-2\hat{S})}{\rho_2}\right]$$

$$= \frac{-2(1+\rho_2^2-2\hat{S})}{\rho_1\rho_2} - \frac{2}{r_1\rho_1\rho_2}\left[\frac{\hat{T}}{r_1} + \frac{\hat{S}}{r_2}\right] ,$$

that is

$$\frac{2\hat{a}_2}{(1+\hat{a}_2)^2\rho_1}\left[\frac{(1+\rho_1^2-2\hat{T})}{\rho_1} + \frac{(1+\rho_2-2\hat{S})}{\rho_2}\right]$$

$$= \frac{(1+\rho_2^2-2\hat{S})}{\rho_1\rho_2} + \frac{1}{r_1\rho_1\rho_2}\left[\frac{\hat{T}}{r_1} + \frac{\hat{S}}{r_2}\right] , \qquad (2.15)$$

where $\hat{a}_2 \equiv a_2(\hat{T},\hat{S})$.

Similarly, differentiating a_2 with respect to S (holding T constant) and setting $\frac{\partial a_2}{\partial S} = 0$, we obtain

$$\frac{2\hat{a}_2}{(1+\hat{a}_2)^2\rho_2}\left[\frac{(1+\rho_1^2-2\hat{T})}{\rho_1}+\frac{(1+\rho_2^2-2\hat{S})}{\rho_2}\right]$$

$$=\frac{(1+\rho_1^2-2\hat{T})}{\rho_1\rho_2}+\frac{1}{r_2\rho_1\rho_2}\left[\frac{\hat{T}}{r_1}+\frac{\hat{S}}{r_2}\right]. \qquad (2.16)$$

From (2.15) and (2.16) we solve for $\hat{T}$ and $\hat{S}$ simultaneously as follows: expressing $\hat{T}$ in terms of $\hat{S}$ using (2.15), we may write

$$\hat{T}\left[\frac{-4\hat{a}_2}{(1+\hat{a}_2)^2\rho_1^2}-\frac{1}{r_1^2\rho_1\rho_2}\right]=\frac{(1+\rho_2^2-2\hat{S})}{\rho_1\rho_2}+\frac{\hat{S}}{r_1r_2\rho_1\rho_2}$$

$$-\frac{2\hat{a}_2(1+\rho_1^2)}{(1+\hat{a}_2)^2\rho_1^2}-\frac{2\hat{a}_2(1+\rho_2^2-2\hat{S})}{(1+\hat{a}_2)^2\rho_1\rho_2}$$

giving

$$\hat{T}=\left[\frac{4\hat{a}_2}{(1+\hat{a}_2)^2\rho_1^2}+\frac{1}{r_1^2\rho_1\rho_2}\right]^{-1}\left[\frac{-\hat{S}}{\rho_1\rho_2}\left(\frac{4\hat{a}_2}{(1+\hat{a}_2)^2}+\frac{1}{r_1r_2}-2\right)\right.$$

$$\left.+\frac{2\hat{a}_2}{(1+\hat{a}_2)^2\rho_1}\left(\frac{1+\rho_1^2}{\rho_1}+\frac{1+\rho_2^2}{\rho_2}\right)-\frac{(1+\rho_2^2)}{\rho_1\rho_2}\right],$$

and similarly, expressing $\hat{S}$ in terms of $\hat{T}$ in (2.16), we may write

$$\hat{S}=\left[\frac{4\hat{a}_2}{(1+\hat{a}_2)^2\rho_2^2}+\frac{1}{r_2^2\rho_1\rho_2}\right]^{-1}\left[\frac{-\hat{T}}{\rho_1\rho_2}\left(\frac{4\hat{a}_2}{(1+\hat{a}_2)^2}+\frac{1}{r_1r_2}-2\right)\right.$$

$$\left.+\frac{2\hat{a}_2}{(1+\hat{a}_2)^2\rho_2}\left(\frac{1+\rho_1^2}{\rho_1}+\frac{1+\rho_2^2}{\rho_2}\right)-\frac{(1+\rho_1^2)}{\rho_1\rho_2}\right].$$

Substituting the expression for $\hat{T}$ in the expression for $\hat{S}$, we get

$$\hat{S} = \left[\frac{4\hat{a}_2}{(1+\hat{a}_2)^2\rho_2^2} + \frac{1}{r_2^2\rho_1\rho_2}\right]^{-1}\left\{\frac{2\hat{a}_2}{(1+\hat{a}_2)^2\rho_2}\left(\frac{1+\rho_1^2}{\rho_1} + \frac{1+\rho_2^2}{\rho_2}\right) - \frac{(1+\rho_1^2)}{\rho_1\rho_2}\right.$$

$$- \frac{1}{\rho_1\rho_2}\left(\frac{4\hat{a}_2}{(1+\hat{a}_2)^2} + \frac{1}{r_1r_2} - 2\right)\left[\frac{4\hat{a}_2}{(1+\hat{a}_2)^2\rho_1^2} + \frac{1}{r_1^2\rho_1\rho_2}\right]^{-1}$$

$$\times \left[\frac{2\hat{a}_2}{(1+\hat{a}_2)^2\rho_1}\left(\frac{1+\rho_1^2}{\rho_1} + \frac{1+\rho_2^2}{\rho_2}\right) - \frac{(1+\rho_2^2)}{\rho_1\rho_2}\right.$$

$$\left.\left. - \frac{\hat{S}}{\rho_1\rho_2}\left(\frac{4\hat{a}_2}{(1+\hat{a}_2)^2} + \frac{1}{r_1r_2} - 2\right)\right]\right\} ,$$

which becomes

$$\hat{S} = \left\{\left[\frac{2\hat{a}_2}{(1+\hat{a}_2)^2\rho_2}\left(\frac{1+\rho_1^2}{\rho_1} + \frac{1+\rho_2^2}{\rho_2}\right) - \frac{(1+\rho_1^2)}{\rho_1\rho_2}\right]\left[\frac{4\hat{a}_2}{(1+\hat{a}_2)^2\rho_1^2} + \frac{1}{r_1^2\rho_1\rho_2}\right]\right.$$

$$\left. - \frac{1}{\rho_1\rho_2}\left[\frac{4\hat{a}_2}{(1+\hat{a}_2)^2} + \frac{1}{r_1r_2} - 2\right]\left[\frac{2\hat{a}_2}{(1+\hat{a}_2)^2\rho_1}\left(\frac{1+\rho_1^2}{\rho_1} + \frac{1+\rho_2^2}{\rho_2}\right) - \frac{(1+\rho_2^2)}{\rho_1\rho_2}\right]\right\}$$

$$\times \left\{\left[\frac{4\hat{a}_2}{(1+\hat{a}_2)^2\rho_2^2} + \frac{1}{r_2^2\rho_1\rho_2}\right]\left[\frac{4\hat{a}_2}{(1+\hat{a}_2)^2\rho_1^2} + \frac{1}{r_1^2\rho_1\rho_2}\right]\right.$$

$$\left. - \frac{1}{\rho_1^2\rho_2^2}\left[\frac{4\hat{a}_2}{(1+\hat{a}_2)^2} + \frac{1}{r_1r_2} - 2\right]^2\right\}^{-1} .$$

The numerator of $\hat{S}$ is

$$\left[\frac{8\hat{a}_2^2}{(1+\hat{a}_2)^4\rho_1^2\rho_2}\left(\frac{1+\rho_1^2}{\rho_1} + \frac{1+\rho_2^2}{\rho_2}\right) + \frac{2\hat{a}_2}{(1+\hat{a}_2)^2r_1^2\rho_1\rho_2^2}\left(\frac{1+\rho_1^2}{\rho_1} + \frac{1+\rho_2^2}{\rho_2}\right)\right.$$

$$- \frac{4\hat{a}_2(1+\rho_1^2)}{(1+\hat{a}_2)^2\rho_1^3\rho_2} - \frac{(1+\rho_1^2)}{r_1^2\rho_1^2\rho_2^2} - \frac{8\hat{a}_2^2}{(1+\hat{a}_2)^4\rho_1^2\rho_2}\left(\frac{1+\rho_1^2}{\rho_1} + \frac{1+\rho_2^2}{\rho_2}\right)$$

continued

$$- \frac{2\hat{a}_2}{(1+\hat{a}_2)^2 r_1 r_2 \rho_1^2 \rho_2} \left(\frac{1+\rho_1^2}{\rho_1} + \frac{1+\rho_2^2}{\rho_2} \right) + \frac{4\hat{a}_2}{(1+\hat{a}_2)^2 \rho_1^2 \rho_2} \left(\frac{1+\rho_1^2}{\rho_1} + \frac{1+\rho_2^2}{\rho_2} \right)$$

$$+ \frac{4\hat{a}_2(1+\rho_2^2)}{(1+\hat{a}_2)^2 \rho_1^2 \rho_2^2} + \frac{(1+\rho_2^2)}{r_1 r_2 \rho_1^2 \rho_2^2} - \frac{2(1+\rho_2^2)}{\rho_1^2 \rho_2^2} \Bigg]$$

$$= [(1+\hat{a}_2)^2 \, r_1^2 r_2^2 \, \rho_1^3 \, \rho_2^3]^{-1}$$

$$\times \Big[2\hat{a}_2(1+\rho_1^2) r_2^2 \rho_1 \rho_2 + 2\hat{a}_2(1+\rho_2^2) r_2^2 \rho_1^2 - (1+\hat{a}_2^2+2\hat{a}_2)(1+\rho_1^2) r_2^2 \rho_1 \rho_2$$

$$-2\hat{a}_2(1+\rho_1^2) r_1 r_2 \rho_2^2 - 2\hat{a}_2(1+\rho_2^2) r_1 r_2 \rho_1 \rho_2 + 8\hat{a}_2(1+\rho_2^2) r_1^2 r_2^2 \rho_1 \rho_2$$

$$+(1+\hat{a}_2^2+2\hat{a}_2)(1+\rho_2^2) r_1 r_2 \rho_1 \rho_2 - 2(1+\hat{a}_2^2+2\hat{a}_2)(1+\rho_2^2) r_1^2 r_2^2 \rho_1 \rho_2 \Big]$$

$$= [(1+\hat{a}_2)^2 r_1^2 r_2^2 \rho_1^3 \rho_2^3]^{-1}$$

$$\times \Big\{ 2\hat{a}_2 [(1+\rho_2^2) r_2^2 \rho_1^2 - (1+\rho_1^2) r_1 r_2 \rho_2^2 + 2(1+\rho_2^2) r_1^2 r_2^2 \rho_1 \rho_2]$$

$$-(1+\hat{a}_2^2) \rho_1 \rho_2 [(1+\rho_1^2) r_2^2 - (1+\rho_2^2) r_1 r_2 + 2(1+\rho_2^2) r_1^2 r_2^2] \Big\} ,$$

and the denominator of $\hat{S}$ is

$$\Bigg[\frac{16\hat{a}_2^2}{(1+\hat{a}_2)^4 \rho_1^2 \rho_2^2} + \frac{4\hat{a}_2}{(1+\hat{a}_2)^2 r_1^2 \rho_1 \rho_2^3} + \frac{4\hat{a}_2}{(1+\hat{a}_2)^2 r_2^2 \rho_1^3 \rho_2} + \frac{1}{r_1^2 r_2^2 \rho_1 \rho_2}$$

$$- \frac{16\hat{a}_2^2}{(1+\hat{a}_2)^4 \rho_1^2 \rho_2^2} - \frac{1}{r_1^2 r_2^2 \rho_1^2 \rho_2^2} - \frac{4}{\rho_1^2 \rho_2^2} - \frac{8\hat{a}_2}{(1+\hat{a}_2)^2 r_1 r_2 \rho_1^2 \rho_2^2}$$

$$+ \frac{16\hat{a}_2}{(1+\hat{a}_2)^2 \rho_1^2 \rho_2^2} + \frac{4}{r_1 r_2 \rho_1^2 \rho_2^2} \Bigg]$$

$$= 4[(1+\hat{a}_2)^2 r_1^2 r_2^2 \rho_1^3 \rho_2^3 \,]^{-1}$$

continued

$$\times [\hat{a}_2 r_2^2\rho_1^2+\hat{a}_2 r_1^2\rho_2^2 - (1+\hat{a}_2^2+2\hat{a}_2)r_1^2r_2^2\rho_1\rho_2 - 2\hat{a}_2 r_1 r_2\rho_1\rho_2$$

$$+ 4\hat{a}_2 r_1^2 r_2^2\rho_1\rho_2 + (1+\hat{a}_2^2+2\hat{a}_2)r_1 r_2\rho_1\rho_2]$$

$$= 4[(1+\hat{a}_2)^2 r_1^2 r_2^2\rho_1^3\rho_2^3]^{-1}$$

$$\times \{\hat{a}_2[r_1^2\rho_2^2+r_2^2\rho_1^2+2r_1^2r_2^2\rho_1\rho_2] + (1+\hat{a}_2^2)r_1 r_2\rho_1\rho_2[1-r_1 r_2]\} \ .$$

Finally from (2.15) and (2.16), solving for $\hat{S}$, we find

$$\hat{S} = \frac{1}{4}\Big[\hat{a}_2(r_1^2\rho_2^2+r_2^2\rho_1^2+2r_1^2r_2^2\rho_1\rho_2) + (1+\hat{a}_2^2)r_1 r_2\rho_1\rho_2(1-r_1 r_2)\Big]^{-1}$$

$$\times\Big\{2\hat{a}_2[r_2^2\rho_1^2(1+\rho_2^2) - r_1 r_2\rho_2^2(1+\rho_1^2) + 2r_1^2 r_2^2\rho_1\rho_2(1+\rho_2^2)]$$

$$- (1+\hat{a}_2^2)\rho_1\rho_2[r_2^2(1+\rho_1^2) - r_1 r_2(1+\rho_2^2) + 2r_1^2 r_2^2(1+\rho_2^2)]\Big\} \qquad (2.17)$$

and similarly, solving for $\hat{T}$, we find (by symmetry)

$$\hat{T} = \frac{1}{4}\Big[\hat{a}_2(r_1^2\rho_2^2+r_2^2\rho_1^2+2r_1^2r_2^2\rho_1\rho_2) + (1+\hat{a}_2^2)r_1 r_2\rho_1\rho_2(1-r_1 r_2)\Big]^{-1}$$

$$\times\Big\{2\hat{a}_2[r_1^2\rho_2^2(1+\rho_1^2) - r_1 r_2\rho_1^2(1+\rho_2^2) + 2r_1^2 r_2^2\rho_1\rho_2(1+\rho_1^2)]$$

$$- (1+\hat{a}_2^2)\rho_1\rho_2[r_1^2(1+\rho_2^2) - r_1 r_2(1+\rho_1^2) + 2r_1^2 r_2^2(1+\rho_1^2)]\Big\} \ . \qquad (2.18)$$

Denote the deonominator in (2.17) and (2.18) by $D(\hat{a}_2,r_1,r_2)$.

Then

$$D(\hat{a}_2,r_1,r_2) = 4[\hat{a}_2(r_1^2\rho_2^2+r_2^2\rho_1^2+2r_1^2r_2^2\rho_1\rho_2) + (1+\hat{a}_2^2)r_1 r_2\rho_1\rho_2(1-r_1 r_2)]$$

$$= 4[\hat{a}_2(r_1\rho_2+r_2\rho_1)^2 - 2\hat{a}_2 r_1 r_2\rho_1\rho_2(1-r_1 r_2)$$

$$+ (1+\hat{a}_2^2)r_1 r_2\rho_1\rho_2(1-r_1 r_2)]$$

$$= 4[(1-\hat{a}_2)^2 r_1 r_2\rho_1\rho_2(1-r_1 r_2) + \hat{a}_2(r_1\rho_2+r_2\rho_1)^2] \ . \qquad (2.19)$$

With this notation and using (2.18), we have

$$\frac{1+\rho_1^2-2\hat{T}}{\rho_1} = \frac{1}{D(\hat{a}_2,r_1,r_2)\rho_1}$$

$$\times[4\hat{a}_2r_1^2\rho_2^2(1+\rho_1^2)+4\hat{a}_2r_2^2\rho_1^2(1+\rho_1^2)+8\hat{a}_2r_1^2r_2^2\rho_1\rho_2(1+\rho_1^2)$$

$$+ 4(1+\hat{a}_2^2)r_1r_2\rho_1\rho_2(1+\rho_1^2) - 4(1+\hat{a}_2^2)r_1^2r_2^2\rho_1\rho_2(1+\rho_1^2)$$

$$- 4\hat{a}_2r_1^2\rho_2^2(1+\rho_1^2) + 4\hat{a}_2r_1r_2\rho_1^2(1+\rho_2^2)$$

$$- 8\hat{a}_2r_1^2r_2^2\rho_1\rho_2(1+\rho_1^2) + 2(1+\hat{a}_2^2)r_1^2\rho_1\rho_2(1+\rho_2^2)$$

$$- 2(1+\hat{a}_2^2)r_1r_2\rho_1\rho_2(1+\rho_1^2) + 4(1+\hat{a}_2^2)r_1^2r_2^2\rho_1\rho_2(1+\rho_1^2)] ,$$

that is,

$$\frac{1+\rho_1^2-2\hat{T}}{\rho_1} = \frac{2}{D(\hat{a}_2,r_1,r_2)}$$

$$\times\{2\hat{a}_2[r_2^2\rho_1(1+\rho_1^2) + r_1r_2\rho_1(1+\rho_2^2)$$

$$+(1+\hat{a}_2^2)[r_1r_2\rho_2(1+\rho_1^2) + r_1^2\rho_2(1+\rho_2^2)]\}$$

$$= \frac{2[r_2(1+\rho_1^2)+r_1(1+\rho_2^2)][2\hat{a}_2r_2\rho_1+(1+\hat{a}_2^2)r_1\rho_2]}{D(\hat{a}_2,r_1,r_2)} . \tag{2.20}$$

Similarly, from (2.17), we have

$$\frac{1+\rho_2^2-2\hat{S}}{\rho_2} = \frac{2[r_1(1+\rho_2^2)+r_2(1+\rho_1^2)][2\hat{a}_2r_1\rho_2+(1+\hat{a}_2^2)r_2\rho_1]}{D(\hat{a}_2,r_1, r_2)} . \tag{2.21}$$

Using (2.20) and (2.21)

$$\frac{\hat{a}_2}{(1+\hat{a}_2)^2} \left[\frac{(1+\rho_1^2-2\hat{T})}{\rho_1} + \frac{(1+\rho_2^2-2\hat{S})}{\rho_2}\right]^2$$

continued

$$
= \frac{\hat{a}_2}{(1+\hat{a}_2)^2} \cdot \frac{4}{[D(\hat{a}_2,r_1,r_2)]^2} \left[r_1(1+\rho_2^2)+r_2(1+\rho_1^2)\right]^2 \left[r_1\rho_2+r_2\rho_1)(1+\hat{a}_2)^2\right]^2
$$

$$
= \frac{4\hat{a}_2(1+\hat{a}_2)^2(r_1\rho_2+r_2\rho_1)^2\left[r_1(1+\rho_2^2)+r_2(1+\rho_1^2)\right]^2}{[D(\hat{a}_2,r_1,r_2)]^2} , \qquad (2.22)
$$

$$
\frac{(1+\rho_1^2-2\hat{T})(1+\rho_2^2-2\hat{S})}{\rho_1\rho_2}
$$

$$
= \frac{4[r_1(1+\rho_2^2)+r_2(1+\rho_1^2)]^2}{[D(\hat{a}_2,r_1,r_2)]^2} [2\hat{a}_2r_2\rho_1+(1+\hat{a}_2^2)r_1\rho_2][2\hat{a}_2r_1\rho_2+(1+\hat{a}_2^2)r_2\rho_1]
$$

$$
= \frac{4[r_1(1+\rho_2^2)+r_2(1+\rho_1^2)]^2}{[D(\hat{a}_2,r_1,r_2)]^2}
$$

$$
\times\left[(1-\hat{a}_2)^4r_1r_2\rho_1\rho_2+4\hat{a}_2^2(r_1\rho_2+r_2\rho_1)^2 + 2\hat{a}_2(1-\hat{a}_2)^2(r_1\rho_2+r_2\rho_1)^2\right]
$$

$$
= \frac{4[r_1(1+\rho_2^2)+r_2(1+\rho_1^2)]^2}{[D(\hat{a}_2,r_1,r_2)]^2}
$$

$$
\times\left\{(1-\hat{a}_2)^4r_1r_2\rho_1\rho_2 + 2\hat{a}_2(r_1\rho_2+r_2\rho_1)^2[(1+\hat{a}_2)^2-2\hat{a}_2]\right\} \qquad (2.23)
$$

and

$$
\frac{\hat{T}}{r_1} + \frac{\hat{S}}{r_2} = [D(\hat{a}_2,r_1,r_2)r_1r_2]^{-1}
$$

$$
\times\Big\{2\hat{a}_2[r_1^2r_2\rho_2^2(1+\rho_1^2) - r_1r_2^2\rho_1^2(1+\rho_2^2) + 2r_1^2r_2^3\rho_1\rho_2(1+\rho_1^2)]
$$

$$
-(1+\hat{a}_2^2)\rho_1\rho_2[r_1^2r_2(1+\rho_2^2) - r_1r_2^2(1+\rho_1^2) + 2r_1^2r_2^3(1+\rho_1^2)]
$$

$$
+ 2\hat{a}_2[r_1r_2^2\rho_1^2(1+\rho_2^2)-r_1^2r_2\rho_2^2(1+\rho_1^2) + 2r_1^3r_2^2\rho_1\rho_2(1+\rho_2^2)]
$$

$$
- (1+\hat{a}_2^2)\rho_1\rho_2[r_1r_2^2(1+\rho_1^2)-r_1^2r_2(1+\rho_2^2) + 2r_1^3r_2^2(1+\rho_2^2)]\Big\}
$$

$$
= [D(\hat{a}_2,r_1,r_2)r_1r_2]^{-1} \Big\{4\hat{a}_2r_1^2r_2^2\rho_1\rho_2[r_2(1+\rho_1^2)+r_1(1+\rho_2^2)]
$$

$$
-2(1+\hat{a}_2^2)r_1^2r_2^2\rho_1\rho_2[r_2(1+\rho_1^2)+r_1(1+\rho_2^2)]\Big\} ,
$$

or

$$\frac{\hat{T}}{r_1} + \frac{\hat{S}}{r_2} = \frac{-2r_1r_2\rho_1\rho_2(1-\hat{a}_2)^2[r_1(1+\rho_2^2)+r_2(1+\rho_1^2)]}{D(\hat{a}_2,r_1,r_2)} . \tag{2.24}$$

Recalling that $\hat{a}_2 \equiv a_2(\hat{T},\hat{S})$, (2.14) may be written as

$$\frac{1+\hat{a}_2^2}{\hat{a}_2} - 2 = \frac{(1+\rho_1^2-2\hat{T})(1+\rho_2^2-2\hat{S})}{\rho_1\rho_2} - \frac{1}{\rho_1\rho_2}\left[\frac{\hat{T}}{r_1}+\frac{\hat{S}}{r_2}\right]^2$$

$$- \frac{\hat{a}_2}{(1+\hat{a}_2)^2}\left[\frac{(1+\rho_1^2-2\hat{T})}{\rho_1} + \frac{(1+\rho_2^2-2\hat{S})}{\rho_2}\right]^2 .$$

Now substituting (2.22), (2.23) and (2.24) in the above expression , we get

$$\frac{1+\hat{a}_2^2}{\hat{a}_2} - 2 = \frac{4[r_1(1+\rho_2^2)+r_2(1+\rho_1^2)]^2}{[D(\hat{a}_2,r_1,r_2)]^2}$$

$$\times\Big\{(1-\hat{a}_2)^4 r_1r_2\rho_1\rho_2 + 2\hat{a}_2(1+\hat{a}_2)^2(r_1\rho_2+r_2\rho_1)^2$$

$$- 4\hat{a}_2^2(r_1\rho_2+r_2\rho_1)^2 - (1-\hat{a}_2)^4 r_1^2r_2^2\rho_1\rho_2$$

$$- \hat{a}_2(1+\hat{a}_2)^2(r_1\rho_2+r_2\rho_1)^2\Big\}$$

$$= \frac{4[r_1(1+\rho_2^2)+r_2(1+\rho_1^2)]^2}{[D(\hat{a}_2,r_1,r_2]^2}\Big[(1-\hat{a}_2)^4 r_1r_2\rho_1\rho_2(1-r_1r_2)$$

$$+ \hat{a}_2(r_1\rho_2+r_2\rho_1)^2(1-\hat{a}_2)^2\Big]$$

$$= \frac{4[r_1(1+\rho_2^2)+r_2(1+\rho_1^2)]^2(1-\hat{a}_2)^2}{[D(\hat{a}_2,r_1,r_2)]^2}$$

$$\times\Big[(1-\hat{a}_2)^2 r_1r_2\rho_1\rho_2(1-r_1r_2) + \hat{a}_2(r_1\rho_2+r_2\rho_1)^2\Big]$$

and hence,

$$\frac{1+\hat{a}_2^2-2\hat{a}_2}{\hat{a}_2} = \frac{[r_1(1+\rho_2^2)+r_2(1+\rho_1^2)]^2(1-\hat{a}_2)^2}{D(\hat{a}_2,r_1,r_2)} \quad .$$

Thus

$$\frac{D(\hat{a}_2,r_1,r_2)}{\hat{a}_2} = [r_1(1+\rho_2^2)+r_2(1+\rho_1^2)]^2 \tag{2.25}$$

and using (2.19), this is

$$4(1-\hat{a}_2)^2 r_1 r_2 \rho_1 \rho_2 (1-r_1 r_2) + 4\hat{a}_2(r_1\rho_2+r_2\rho_1)^2$$
$$= \hat{a}_2[r_1(1+\rho_2^2) + r_2(1+\rho_1^2)]^2 \quad .$$

To obtain $\hat{a}_2$ we solve the quadratic equation,

$$4\hat{a}_2^2 r_1 r_2 \rho_1 \rho_2 (1-r_1 r_2)$$
$$+\hat{a}_2\left\{4(r_1\rho_2+r_2\rho_1)^2 - 8r_1r_2\rho_1\rho_2(1-r_1r_2)-[r_1(1+\rho_2^2)+r_2(1+\rho_1^2)]^2\right\}$$
$$+ 4r_1r_2\rho_1\rho_2(1-r_1r_2) = 0 \quad .$$

Putting this equation into a more convenient form, we get

$$4\hat{a}_2^2 r_1 r_2 \rho_1 \rho_2 (1-r_1 r_2) + 4r_1r_2\rho_1\rho_2(1-r_1r_2)$$
$$-\hat{a}_2\Big\{[r_1(1+\rho_2^2)+r_2(1+\rho_1^2) - 2(r_1\rho_2+r_2\rho_1)][r_1(1+\rho_2^2)+r_2(1+\rho_1^2)$$
$$+ 2(r_1\rho_2+r_2\rho_1)]$$
$$+ 8r_1r_2\rho_1\rho_2(1-r_1r_2)\Big\} = 0$$

or

$$4\hat{a}_2^2 r_1 r_2 \rho_1 \rho_2 (1-r_1 r_2) + 4r_1r_2\rho_1\rho_2(1-r_1r_2)$$
$$-\hat{a}_2\Big\{[r_1(1-\rho_2)^2 + r_2(1-\rho_1)^2][r_1(1+\rho_2)^2 + r_2(1+\rho_1)^2]$$
$$+ 8r_1r_2\rho_1\rho_2(1-r_1r_2)\Big\} = 0 \quad .$$

Finally, the value of $\hat{a}_2$ is

$$\hat{a}_2 = \frac{8r_1r_2\rho_1\rho_2(1-r_1r_2)+[r_1(1-\rho_2)^2+r_2(1-\rho_1)^2][r_1(1+\rho_2)^2+r_2(1+\rho_1)^2]}{8r_1r_2\rho_1\rho_2(1-r_1r_2)}$$

$$- [8r_1r_2\rho_1\rho_2(1-r_1r_2)]^{-1}$$

$$\times\Big[\{8r_1r_2\rho_1\rho_2(1-r_1r_2)+[r_1(1-\rho_2)^2+r_2(1-\rho_1)^2][r_1(1+\rho_2)^2+r_2(1+\rho_1)^2]\}^2$$

$$- 64r_1^2r_2^2\rho_1^2\rho_2^2(1-r_1r_2)^2\Big]^{\frac{1}{2}}$$

$$= 1 + \frac{[r_1(1-\rho_2)^2+r_2(1-\rho_1)^2][r_1(1+\rho_2)^2+r_2(1+\rho_1)^2]}{8r_1r_2\rho_1\rho_2(1-r_1r_2)}$$

$$-\Bigg[\frac{[r_1(1-\rho_2)^2+r_2(1-\rho_1)^2]^2[r_1(1+\rho_2)^2+r_2(1+\rho_1)^2]^2}{64r_1^2r_2^2\rho_1^2\rho_2^2(1-r_1r_2)^2}$$

$$+ \frac{2[r_1(1-\rho_2)^2+r_2(1-\rho_1)^2][r_1(1+\rho_2)^2+r_2(1+\rho_1)^2]}{8r_1r_2\rho_1\rho_2(1-r_1r_2)}\Bigg]^{\frac{1}{2}} ,$$

which may be written as

$$\hat{a}_2 = 1 + V - (V^2 + 2V)^{\frac{1}{2}} , \tag{2.26}$$

where

$$V = \frac{[r_1(1-\rho_2)^2+r_2(1-\rho_1)^2][r_1(1+\rho_2)^2+r_2(1+\rho_1)^2]}{8r_1r_2\rho_1\rho_2(1-r_1r_2)} .$$

Using the relation (2.25), we may further reduce (2.20), (2.21) and (2.24) as follows:

$$\left.\begin{aligned}
\frac{1+\rho_1^2-2\hat{T}}{\rho_1} &= \frac{2[2\hat{a}_2 r_2\rho_1+(1+\hat{a}_2^2)r_1\rho_2]}{\hat{a}_2[r_1(1+\rho_2^2)+r_2(1+\rho_1^2)]} \\
&= \frac{2[(1-\hat{a}_2)^2 r_1\rho_2+2\hat{a}_2(r_1\rho_2+r_2\rho_1)]}{\hat{a}_2[r_1(1+\rho_2^2)+r_2(1+\rho_1^2)]}\ , \\
\frac{1+\rho_2^2-2\hat{S}}{\rho_2} &= \frac{2[(1-\hat{a}_2)^2 r_2\rho_1+2\hat{a}_2(r_1\rho_2+r_2\rho_1)]}{\hat{a}_2[r_1(1+\rho_2^2)+r_2(1+\rho_1^2)]}\ , \\
\frac{\hat{T}}{r_1}+\frac{\hat{S}}{r_2} &= \frac{-2r_1r_2\rho_1\rho_2(1-\hat{a}_2)^2}{\hat{a}_2[r_1(1+\rho_2^2)+r_2(1+\rho_1^2)]}\ ,
\end{aligned}\right\} \tag{2.27}$$

and thus

$$\begin{aligned}
\hat{a}_1 &= \frac{-\hat{a}_2}{1+\hat{a}_2}\left[\frac{(1+\rho_1^2-2\hat{T})}{\rho_1}+\frac{(1+\rho_2^2-2\hat{S})}{\rho_2}\right] \\
&= \frac{-2[(1-\hat{a}_2)^2(r_1\rho_2+r_2\rho_1)+4\hat{a}_2(r_1\rho_2+r_2\rho_1)]}{(1+\hat{a}_2)[r_1(1+\rho_2^2)+r_2(1+\rho_1^2)]} \\
&= \frac{-2(1+\hat{a}_2)(r_1\rho_2+r_2\rho_1)}{[r_1(1+\rho_2^2)+r_2(1+\rho_1^2)]}\ .
\end{aligned} \tag{2.28}$$

With $v = 0$ and using (2.8), (2.9) and (2.11), $\hat{\beta}_{11}$, $\hat{\beta}_{12}$ and $\hat{\beta}_{21}$ are:

$$\begin{aligned}
\hat{\beta}_{11} &= \frac{(1-2\hat{T})(1-2\hat{S})}{\rho_1\rho_2}+1-\frac{1}{\rho_1\rho_2}\left[\frac{\hat{T}}{r_1}+\frac{\hat{S}}{r_2}\right]^2 \\
&= \frac{(1+\rho_1^2-2\hat{T})(1+\rho_2^2-2\hat{S})}{\rho_1\rho_2}+2-\frac{1}{\rho_1\rho_2}\left[\frac{\hat{T}}{r_1}+\frac{\hat{S}}{r_2}\right]^2 \\
&\quad -\left[1+\frac{\rho_1}{\rho_2}(1+\rho_2^2-2\hat{S})+\frac{\rho_2}{\rho_1}(1+\rho_1^2-2\hat{T})-\rho_1\rho_2\right] \\
&= \frac{1+\hat{a}_1^2+\hat{a}_2^2}{\hat{a}_2}-\left[1+\frac{\rho_1}{\rho_2}(1+\rho_2^2-2\hat{S})+\frac{\rho_2}{\rho_1}(1+\rho_1^2-2\hat{T})-\rho_1\rho_2\right],
\end{aligned}$$

$$\hat{\beta}_{12} = -\frac{(1-2\hat{T})}{\rho_1} - \frac{(1+\rho_2^2-2\hat{S})}{\rho_2}$$

$$= \frac{-(1+\rho_1^2-2\hat{T})}{\rho_1} - \frac{(1+\rho_2^2-2\hat{S})}{\rho_2} + \rho_1$$

$$= \frac{\hat{a}_1(1+\hat{a}_2)}{\hat{a}_2} + \rho_1$$

and

$$\hat{\beta}_{21} = \frac{\hat{a}_1(1+\hat{a}_2)}{\hat{a}_2} + \rho_2 \quad .$$

Substituting the above expressions in $\hat{a}_2(\hat{\beta}_{11}-\hat{a}_2) - \hat{a}_2^2(\hat{\beta}_{12}-\hat{a}_1)(\hat{\beta}_{21}-\hat{a}_1)$, we get

$$\hat{a}_2(\hat{\beta}_{11}-\hat{a}_2) - \hat{a}_2^2(\hat{\beta}_{12}-\hat{a}_1)(\hat{\beta}_{21}-\hat{a}_1)$$

$$= (1+\hat{a}_1^2)-\hat{a}_2\left[1 + \frac{\rho_1}{\rho_2}(1+\rho_2^2-2\hat{S})+\frac{\rho_2}{\rho_1}(1+\rho_1^2-2\hat{T})-\rho_1\rho_2\right]$$

$$-(\hat{a}_1+\hat{a}_2\rho_1)(\hat{a}_1+\hat{a}_2\rho_2)$$

$$= 1+\hat{a}_1^2-\hat{a}_2(1-\rho_1\rho_2)-\hat{a}_1^2-\hat{a}_1\hat{a}_2(\rho_1+\rho_2)-\hat{a}_2^2\rho_1\rho_2$$

$$-\hat{a}_2\left[\frac{\rho_1}{\rho_2}(1+\rho_2^2-2\hat{S})+\frac{\rho_2}{\rho_1}(1+\rho_1^2-2\hat{T})\right]$$

$$= (1-\hat{a}_2)(1+\hat{a}_2\rho_1\rho_2)-\hat{a}_2\left[\hat{a}_1(\rho_1+\rho_2)+ \frac{\rho_1}{\rho_2}(1+\rho_2^2-2\hat{S})+ \frac{\rho_2}{\rho_1}(1+\rho_1^2-2\hat{T})\right] ,$$

and, finally, substituting the relations (2.27) and (2.28), we obtain

$$\hat{a}_2(\hat{\beta}_{11}-\hat{a}_2)-\hat{a}_2^2(\hat{\beta}_{12}-\hat{a}_1)(\hat{\beta}_{21}-\hat{a}_1)$$

$$= (1-\hat{a}_2)(1+\hat{a}_2\rho_1\rho_2) - 2[r_1(1+\rho_2^2)+r_2(1+\rho_1^2)]^{-1}$$

$$\times\left[-\hat{a}_2(1+\hat{a}_2)(r_1\rho_2+r_2\rho_1)\rho_1-\hat{a}_2(1+\hat{a}_2)(r_1\rho_2+r_2\rho_1)\rho_2\right.$$

$$+ (1-\hat{a}_2)^2r_2\rho_1^2+2\hat{a}_2(r_1\rho_2+r_2\rho_1)\rho_1+(1-\hat{a}_2)^2r_1\rho_2^2$$

$$\left.+ 2\hat{a}_2(r_1\rho_2+r_2\rho_1)\rho_2\right]$$

continued

$$= (1-\hat{a}_2)(1+\hat{a}_2\rho_1\rho_2) - 2[r_1(1+\rho_2^2)+r_2(1+\rho_1^2)]^{-1}$$

$$\times\Big[\hat{a}_2(1-\hat{a}_2)(r_1\rho_2+r_2\rho_1)\rho_1+\hat{a}_2(1-\hat{a}_2)(r_1\rho_2+r_2\rho_1)\rho_2$$

$$+ (1-\hat{a}_2)^2(r_1\rho_2^2+r_2\rho_1^2)\Big] \quad .$$

Hence

$$\hat{a}_2(\hat{\beta}_{11}-\hat{a}_2)-\hat{a}_2^2(\hat{\beta}_{12}-\hat{a}_1)(\hat{\beta}_{21}-\hat{a}_1)$$

$$= (1-\hat{a}_2)[r_1(1+\rho_2^2)+r_2(1+\rho_1^2)]^{-1}$$

$$\times\Big[r_1(1+\rho_2^2)+r_2(1+\rho_1^2)+\hat{a}_2r_1\rho_1\rho_2(1+\rho_2^2)+\hat{a}_2r_2\rho_1\rho_2(1+\rho_1^2)$$

$$- 2\hat{a}_2(r_1\rho_2+r_2\rho_1)\rho_1 -2\hat{a}_2(r_1\rho_2+r_2\rho_1)\rho_2 - 2(r_1\rho_2^2+r_2\rho_1^2)$$

$$+ 2\hat{a}_2(r_1\rho_2^2+r_2\rho_1^2)\Big]$$

$$= (1-\hat{a}_2)[r_1(1+\rho_2^2)+r_2(1+\rho_1^2)]^{-1}$$

$$\times[r_1(1-\rho_2^2)+r_2(1-\rho_1^2)-\hat{a}_2\rho_1\rho_2(r_1+r_2-r_1\rho_2^2-r_2\rho_1^2)]$$

$$= \frac{(1-\hat{a}_2)(1-\hat{a}_2\rho_1\rho_2)[r_1(1-\rho_2^2)+r_2(1-\rho_1^2)]}{[r_1(1+\rho_2^2)+r_2(1+\rho_1^2)]} \quad . \tag{2.29}$$

Again using the relations (2.27) and (2.28), we obtain

$$(1-\hat{a}_1+\hat{a}_2) = 1 + \hat{a}_2 + \frac{2(1+\hat{a}_2)(r_1\rho_2+r_2\rho_1)}{[r_1(1+\rho_2^2)+r_2(1+\rho_1^2)}$$

$$= \frac{(1+\hat{a}_2)[r_1(1+\rho_2)^2+r_2(1+\rho_1)^2]}{[r_1(1+\rho_2^2)+r_2(1+\rho_1^2)]}$$

and

$$(1+\hat{a}_1+\hat{a}_2) = \frac{(1+\hat{a}_2)[r_1(1-\rho_2)^2+r_2(1-\rho_1)^2]}{[r_1(1+\rho_2^2)+r_2(1+\rho_1^2)]} \quad . \tag{2.30}$$

Then

$$(1-\hat{a}_1+\hat{a}_2)(1+\hat{a}_1+\hat{a}_2)$$

$$= \frac{(1+\hat{a}_2)^2}{[r_1(1+\rho_2^2)+r_2(1+\rho_1^2)]^2} \cdot \Big\{ r_1^2(1-\rho_2^2)^2+r_2^2(1-\rho_1^2)^2$$

$$+r_1r_2[(1+\rho_2)^2(1-\rho_1)^2+(1+\rho_1)^2(1-\rho_2)^2]\Big\}$$

$$= \frac{(1+\hat{a}_2)^2\Big\{r_1^2(1-\rho_2^2)^2+r_2^2(1-\rho_1^2)^2+2r_1r_2[(\rho_1-\rho_2)^2+(1-\rho_1\rho_2)^2]\Big\}}{[r_1(1+\rho_2^2)+r_2(1+\rho_1^2)]^2}$$

and thus

$$(1-\hat{a}_1+\hat{a}_2)^{\frac{1}{2}}(1+\hat{a}_1+\hat{a}_2)^{\frac{1}{2}}(1-\hat{a}_2)$$

$$= [r_1(1+\rho_2^2)+r_2(1+\rho_1^2)]^{-1}(1+\hat{a}_2)(1-\hat{a}_2)$$

$$\times\Big\{r_1^2(1-\rho_2^2)^2+r_2^2(1-\rho_1^2)^2+2r_1r_2[(\rho_1-\rho_2)^2+(1-\rho_1\rho_2)^2]\Big\}^{\frac{1}{2}} \quad . \qquad (2.31)$$

Finally, using (2.29) and (2.31), we obtain

$$\left.\frac{(1-\hat{a}_1+\hat{a}_2)^{\frac{1}{2}}(1+\hat{a}_1+\hat{a}_2)^{\frac{1}{2}}(1-\hat{a}_2)}{[\hat{a}_2(\hat{\beta}_{11}-\hat{a}_2)-\hat{a}_2^2(\hat{\beta}_{12}-\hat{a}_1)(\hat{\beta}_{21}-\hat{a}_1)]}\right|_{v=0,T=\hat{T},S=\hat{S}}$$

$$= (1-\hat{a}_2\rho_1\rho_2)^{-1}[r_1(1-\rho_2^2)+r_2(1-\rho_1^2)]^{-1}\ (1+\hat{a}_2)$$

$$\times\Big\{r_1^2(1-\rho_2^2)^2+r_2^2(1-\rho_1^2)^2+2r_1r_2[(\rho_1-\rho_2)^2+(1-\rho_1\rho_2)^2]\Big\}^{\frac{1}{2}} \quad . \qquad (2.32)$$

To find $\left.\left(\frac{\partial\hat{a}_2}{\partial v}\right)^2\right|_{v=0}$, we proceed as follows: from (2.8) and (2.11), where now $a_2 \equiv a_2(T,S,v)$, we may write

$$a_2 + \frac{1}{a_2} + \frac{a_2}{(1+a_2)^2}\left[\frac{(1+\rho_1^2-2T)}{\rho_1} + \frac{(1+\rho_2^2-2S)}{\rho_2}\right]^2$$

$$= 2 + \frac{(1+\rho_1^2-2T)(1+\rho_2^2-2S)}{\rho_1\rho_2} - \frac{1}{\rho_1\rho_2}\left[\frac{v(r_1+r_2)}{r_1r_2} - \frac{T}{r_1} - \frac{S}{r_2}\right]^2 . \qquad (2.33)$$

Differentiating (2.33) partially with respect to v and holding T and S constant, it can be seen that

$$\left(1 - \frac{1}{a_2^2}\right)\frac{\partial a_2}{\partial v} + \left[\frac{(1+\rho_1^2-2T)}{\rho_1} + \frac{(1+\rho_2^2-2S)}{\rho_2}\right]^2 \frac{(1-a_2)}{(1+a_2)^3}\frac{\partial a_2}{\partial v}$$

$$= \frac{-2(r_1+r_2)}{r_1 r_2 \rho_1 \rho_2}\left[\frac{v(r_1+r_2)}{r_1 r_2} - \frac{T}{r_1} - \frac{S}{r_2}\right] .$$

Then putting $v = 0$, $T = \hat{T}$ and $S = \hat{S}$ (again $\hat{a}_2 \equiv a_2(\hat{T},\hat{S})$)

$$\left.\frac{\partial a_2}{\partial v}\right|_{v=0,a_2=\hat{a}_2} = \frac{\dfrac{2(r_1+r_2)}{r_1 r_2 \rho_1 \rho_2}\left[-\dfrac{\hat{T}}{r_1} - \dfrac{\hat{S}}{r_2}\right]}{\dfrac{1-\hat{a}_2^2}{\hat{a}_2^2} - \dfrac{(1-\hat{a}_2)}{(1+\hat{a}_2)^3}\left[\dfrac{(1+\rho_1^2-2\hat{T})}{\rho_1} + \dfrac{(1+\rho_2^2-2\hat{S})}{\rho_2}\right]^2} ,$$

and using the relationships (2.27)

$$\left.\frac{\partial \hat{a}_2}{\partial v}\right|_{v=0} = \frac{2(r_1+r_2)\left[\dfrac{2(1-\hat{a}_2)^2}{\hat{a}_2[r_1(1+\rho_2^2)+r_2(1+\rho_1^2)]}\right]}{\dfrac{1-\hat{a}_2^2}{\hat{a}_2^2} - \dfrac{1-\hat{a}_2}{(1+\hat{a}_2)^3}\left[\dfrac{4(1+\hat{a}_2)^4(r_1\rho_2+r_2\rho_1)^2}{\hat{a}_2^2[r_1(1+\rho_2^2)+r_2(1+\rho_1^2)]^2}\right]}$$

$$= \frac{\dfrac{4(r_1+r_2)(1-\hat{a}_2)^2}{\hat{a}_2[r_1(1+\rho_2^2)+r_2(1+\rho_1^2)]}}{\dfrac{1-\hat{a}_2^2}{\hat{a}_2^2}\left[1 - \dfrac{4(r_1\rho_2+r_2\rho_1)^2}{[r_1(1+\rho_2^2)+r_2(1+\rho_1^2)]^2}\right]}$$

$$= 4(r_1+r_2)(1-\hat{a}_2)^2 \cdot \hat{a}_2[r_1(1+\rho_2^2)+r_2(1+\rho_1^2)]$$

$$\times\left[(1-\hat{a}_2)(1+\hat{a}_2)\left\{\left[r_1(1+\rho_2^2)+r_2(1+\rho_1^2)\right]^2 - 4(r_1\rho_2+r_2\rho_1)^2\right]^{-1}\right.$$

continued

$$= \frac{4\hat{a}_2(1-\hat{a}_2)(r_1+r_2)[r_1(1+\rho_2^2)+r_2(1+\rho_1^2)]}{(1+\hat{a}_2)\left\{[r_1(1-\rho_2)^2+r_2(1-\rho_1)^2][r_1(1+\rho_2)^2+r_2(1+\rho_1)^2]\right\}}$$

$$= \frac{4\hat{a}_2(1-\hat{a}_2)(r_1+r_2)[r_1(1+\rho_2^2)+r_2(1+\rho_1^2)]}{(1+\hat{a}_2)\left\{r_1^2(1-\rho_2^2)^2+r_2^2(1-\rho_1^2)^2+2r_1r_2[(\rho_1-\rho_2)^2+(1-\rho_1\rho_2)^2]\right\}} .$$

Thus combining (2.32) and the above, we finally express $\varphi(\hat{T},\hat{S})$ (see (2.14)), as

$$\frac{(1-\hat{a}_1+\hat{a}_2)^{\frac{1}{2}}(1+\hat{a}_1+\hat{a}_2)^{\frac{1}{2}}(1-\hat{a}_2)}{[\hat{a}_2(\hat{\beta}_{11}-\hat{a}_2)-\hat{a}_2^2(\hat{\beta}_{12}-\hat{a}_1)(\hat{\beta}_{21}-\hat{a}_1)]}\left(\frac{\partial\hat{a}_2}{\partial v}\right)^2\Bigg|_{v=0}$$

$$= 16\hat{a}_2^2(1-\hat{a}_2)^2(r_1+r_2)^2[r_1(1+\rho_2^2)+r_2(1+\rho_1^2)]^2$$

$$\times\left\{(1+\hat{a}_2)(1-\hat{a}_2\rho_1\rho_2)[r_1(1-\rho_2^2)+r_2(1-\rho_1^2)]\right\}^{-1}$$

$$\times\left\{r_1^2(1-\rho_2^2)^2+r_2^2(1-\rho_1^2)^2+2r_1r_2[(\rho_1-\rho_2)^2+(1-\rho_1\rho_2)^2]\right\}^{-3/2}$$

(2.34)

With $v = 0$ in equation (2.33) (that is, $a_2 \equiv a_2(T,S)$), partial differentiation with respect to T gives

$$\left(1-\frac{1}{a_2^2}\right)\frac{\partial a_2}{\partial T}+\left[\frac{(1+\rho_1^2-2T)}{\rho_1}+\frac{(1+\rho_2^2-2S)}{\rho_2}\right]^2\frac{(1-a_2)}{(1+a_2)^3}\frac{\partial a_2}{\partial T}$$

$$-\frac{4a_2}{(1+a_2)^2\rho_1}\left[\frac{(1+\rho_1^2-2T)}{\rho_1}+\frac{(1+\rho_2^2-2S)}{\rho_2}\right]$$

$$= \frac{-2(1+\rho_2^2-2S)}{\rho_1\rho_2}-\frac{2}{r_1\rho_1\rho_2}\left[\frac{T}{r_1}+\frac{S}{r_2}\right] , \qquad (2.35)$$

and partial differentiation with respect to S gives

$$\left(1-\frac{1}{a_2^2}\right)\frac{\partial a_2}{\partial S}+\left[\frac{(1+\rho_1^2-2T)}{\rho_1}+\frac{(1+\rho_2^2-2S)^{-2}}{\rho_2}\right]^2\frac{(1-a_2)}{(1+a_2)^3}\frac{\partial a_2}{\partial S}$$

$$-\frac{4a_2}{(1+a_2)^2\rho_2}\left[\frac{(1+\rho_1^2-2T)}{\rho_1}+\frac{(1+\rho_2^2-2S)}{\rho_2}\right]$$

$$=\frac{-2(1+\rho_1^2-2T)}{\rho_1\rho_2}-\frac{2}{r_2\rho_1\rho_2}\left[\frac{T}{r_1}+\frac{S}{r_2}\right] \quad . \qquad (2.36)$$

From (2.35) and (2.36) we obtain the following second-order partial derivatives, where $\frac{\partial a_2}{\partial T}$ and $\frac{\partial a_2}{\partial S}$ are equated to 0, so that $T = \hat{T}$, $S = \hat{S} \Longrightarrow a_2 = \hat{a}_2 \equiv a_2(\hat{T},\hat{S})$:

$$\left(1-\frac{1}{\hat{a}_2^2}\right)\frac{\partial^2 a_2}{\partial T^2}\bigg|_{a_2=\hat{a}_2}+\left[\frac{(1+\rho_1^2-2\hat{T})}{\rho_1}+\frac{(1+\rho_2^2-2\hat{S})}{\rho_2}\right]^2\frac{(1-\hat{a}_2)}{(1+\hat{a}_2)^3}\frac{\partial^2 a_2}{\partial T^2}\bigg|_{a_2=\hat{a}_2}$$

$$+\frac{8\hat{a}_2}{(1+\hat{a}_2)^2\rho_1^2}=\frac{-2}{r_1^2\rho_1\rho_2} \quad ,$$

and hence

$$\frac{\partial^2 a_2}{\partial T^2}\bigg|_{a_2=\hat{a}_2}=\frac{\dfrac{2}{r_1^2\rho_1\rho_2}+\dfrac{8\hat{a}_2}{(1+\hat{a}_2)^2\rho_1^2}}{\dfrac{(1-\hat{a}_2^2)}{\hat{a}_2^2}-\dfrac{(1-\hat{a}_2)}{(1+\hat{a}_2)^3}\left[\dfrac{(1+\rho_1^2-2\hat{T})}{\rho_1}+\dfrac{(1+\rho_2^2-2\hat{S})}{\rho_2}\right]^2}$$

$$=\frac{\dfrac{2}{\rho_1}\left[\dfrac{(1+\hat{a}_2)^2\rho_1+4\hat{a}_2r_1^2\rho_2}{(1+\hat{a}_2)^2r_1^2\rho_1\rho_2}\right]}{\dfrac{(1-\hat{a}_2^2)}{\hat{a}_2^2}\left[1-\dfrac{4(r_1\rho_2+r_2\rho_1)^2}{[r_1(1+\rho_2^2)+r_2(1+\rho_1^2)]^2}\right]}$$

continued

$$= 2\hat{a}_2^2[(1+\hat{a}_2)^2\rho_1 + 4\hat{a}_2 r_1^2\rho_2][r_1(1+\rho_2^2)+r_2(1+\rho_1^2)]^2$$

$$\times\Big[\rho_1(1-\hat{a}_2)(1+\hat{a}_2)^3 r_1^2\rho_1\rho_2\{r_1^2(1-\rho_2^2)^2+r_2^2(1-\rho_1^2)^2$$

$$+ 2r_1 r_2[(\rho_1-\rho_2)^2+(1-\rho_1\rho_2)^2]\}\Big]^{-1}$$

(2.37)

Similarly,

$$\left.\frac{\partial^2 a_2}{\partial S^2}\right|_{a_2=\hat{a}_2} = 2\hat{a}_2^2\left[(1+\hat{a}_2)^2\rho_2 + 4\hat{a}_2 r_2^2\rho_1\right]\left[r_1(1+\rho_2^2)+r_2(1+\rho_1^2)\right]^2$$

$$\times\Big[\rho_2(1-\hat{a}_2)(1+\hat{a}_2)^3 r_2^2\rho_1\rho_2\{r_1^2(1-\rho_2^2)^2+r_2^2(1-\rho_1^2)^2$$

$$+ 2r_1 r_2[(\rho_1-\rho_2)^2+(1-\rho_1\rho_2)^2]\}\Big]^{-1}$$

(2.38)

and

$$\left(1 - \frac{1}{\hat{a}_2^2}\right)\left.\frac{\partial^2 a_2}{\partial T\partial S}\right|_{a_2=\hat{a}_2} + \left[\frac{(1+\rho_1^2-2\hat{T})}{\rho_1} + \frac{(1+\rho_2^2-2\hat{S})}{\rho_2}\right]^2 \frac{(1-\hat{a}_2)}{(1+\hat{a}_2)^3}\left.\frac{\partial^2 a_2}{\partial T\partial S}\right|_{a_2=\hat{a}_2}$$

$$+ \frac{8\hat{a}_2}{(1+\hat{a}_2)^2\rho_1\rho_2} = \frac{4}{\rho_1\rho_2} - \frac{2}{r_1 r_2\rho_1\rho_2} \quad ,$$

so that

$$\left.\frac{\partial^2 a_2}{\partial T \partial S}\right|_{a_2=\hat{a}_2} = \frac{\dfrac{8\hat{a}_2}{(1+\hat{a}_2)^2\rho_1\rho_2} - \dfrac{4}{\rho_1\rho_2} + \dfrac{2}{r_1 r_2 \rho_1 \rho_2}}{\dfrac{(1-\hat{a}_2^2)}{\hat{a}_2^2} - \dfrac{(1-\hat{a}_2)}{(1+\hat{a}_2)^3}\left[\dfrac{(1+\rho_1^2-2\hat{T})}{\rho_1} + \dfrac{(1+\rho_2^2-2\hat{S})}{\rho_2}\right]^2}$$

$$= \frac{\dfrac{2}{\rho_1\rho_2}\left[\dfrac{4\hat{a}_2 r_1 r_2 - 2(1+\hat{a}_2)^2 r_1 r_2 + (1+\hat{a}_2)^2}{(1+\hat{a}_2)^2 r_1 r_2}\right]}{\dfrac{(1-\hat{a}_2^2)}{\hat{a}_2^2}\left[1 - \dfrac{4(r_1\rho_2 + r_2\rho_1)^2}{[r_1(1+\rho_2^2)+r_2(1+\rho_1^2)]^2}\right]}$$

$$= 2\hat{a}_2^2[4\hat{a}_2 r_1 r_2 - 2(1+\hat{a}_2)^2 r_1 r_2 + (1+\hat{a}_2)^2]$$

$$\times [r_1(1+\rho_2^2)+r_2(1+\rho_1^2)]^2[\rho_1\rho_2 r_1 r_2(1-\hat{a}_2)(1+\hat{a}_2)^3]^{-1}$$

$$\times \left\{r_1^2(1-\rho_2^2)^2 + r_2^2(1-\rho_1^2) + 2r_1 r_2[(\rho_1-\rho_2)^2+(1-\rho_1\rho_2)^2]\right\}^{-1} .$$

(2.39)

Thus from (2.36), (2.37) and (2.38)

$$\left.\left[\left(\frac{\partial^2 a_2}{\partial T^2}\right)\left(\frac{\partial^2 a_2}{\partial S^2}\right) - \left(\frac{\partial^2 a_2}{\partial T \partial S}\right)^2\right]^{\frac{1}{2}}\right|_{a_2=\hat{a}_2}$$

$$= 2\hat{a}_2^2[r_1(1+\rho_2^2)+r_2(1+\rho_1^2)]^2[r_1 r_2(\rho_1\rho_2)^{3/2}(1-\hat{a}_2)(1+\hat{a}_2)^3]^{-1}$$

$$\times \left\{r_1^2(1-\rho_2^2)^2 + r_2^2(1-\rho_1^2)^2 + 2r_1 r_2[(\rho_1-\rho_2)^2+(1-\rho_1\rho_2)^2]\right\}^{-1}$$

$$\times \Big\{[(1+\hat{a}_2)^2\rho_1 + 4\hat{a}_2 r_1^2 \rho_2]\,[(1+\hat{a}_2)^2\rho_2 + 4\hat{a}_2 r_2^2 \rho_1]$$

$$- \rho_1\rho_2[4\hat{a}_2 r_1 r_2 - 2(1+\hat{a}_2)^2 r_1 r_2 + (1+\hat{a}_2)^2]^2\Big\}^{\frac{1}{2}} .$$

Simplifying the expression in the last set of brackets above, we see that

$$\Big\{[(1+\hat{a}_2)^2\rho_1+4\hat{a}_2r_1^2\rho_2][(1+\hat{a}_2)^2\rho_2+4\hat{a}_2r_2^2\rho_1]$$

$$-\rho_1\rho_2[4\hat{a}_2r_1r_2-2(1+\hat{a}_2)^2r_1r_2+(1+\hat{a}_2)^2]^2\Big\}^{\frac{1}{2}}$$

$$=\Big\{(1+\hat{a}_2)^4\rho_1\rho_2+16\hat{a}_2^2r_1^2r_2^2\rho_1\rho_2+4\hat{a}_2(1+\hat{a}_2)^2(r_1^2\rho_2^2+r_2^2\rho_1^2)$$

$$-16\hat{a}_2^2r_1^2r_2^2\rho_1\rho_2-4(1+\hat{a}_2)^4r_1^2r_2^2\rho_1\rho_2-(1+\hat{a}_2)^4\rho_1\rho_2$$

$$+16\hat{a}_2(1+\hat{a}_2)^2r_1^2r_2^2\rho_1\rho_2-8\hat{a}_2(1+\hat{a}_2)^2r_1r_2\rho_1\rho_2$$

$$+4(1+\hat{a}_2)^4r_1r_2\rho_1\rho_2\Big\}^{\frac{1}{2}}$$

$$=2(1+\hat{a}_2)\Big\{\hat{a}_2(r_1^2\rho_2^2+r_2^2\rho_1^2)-(1+\hat{a}_2^2+2\hat{a}_2)r_1^2r_2^2\rho_1\rho_2$$

$$+4\hat{a}_2r_1^2r_2^2\rho_1\rho_2-2\hat{a}_2r_1r_2\rho_1\rho_2$$

$$+(1+\hat{a}_2^2+2\hat{a}_2)r_1r_2\rho_1\rho_2\Big\}^{\frac{1}{2}}$$

$$=2(1+\hat{a}_2)\Big[\hat{a}_2(r_1\rho_2+r_2\rho_1)^2+r_1r_2\rho_1\rho_2(1-r_1r_2)(1-\hat{a}_2)^2\Big]^{\frac{1}{2}}.$$

Hence, by (2.19) and (2.25),

$$\Big\{[(1+\hat{a}_2)^2\rho_1+4\hat{a}_2r_1^2\rho_2][(1+\hat{a}_2)^2\rho_2+4\hat{a}_2r_2^2\rho_1]$$

$$-\rho_1\rho_2[4\hat{a}_2r_1r_2-2(1+\hat{a}_2)^2r_1r_2+(1+\hat{a}_2)^2]^2\Big\}^{\frac{1}{2}}$$

$$=\hat{a}_2^{\frac{1}{2}}(1+\hat{a}_2)[r_1(1+\rho_2^2)+r_2(1+\rho_1^2)] \quad .$$

Then

$$\left[\left(\frac{\partial^2 a_2}{\partial T^2}\right)\left(\frac{\partial^2 a_2}{\partial S^2}\right) - \left(\frac{\partial^2 a_2}{\partial T\partial S}\right)^2\right]^{\frac{1}{2}}\Bigg|_{a_2=\hat{a}_2}$$

$$= 2\hat{a}_2^{5/2}[r_1(1+\rho_2^2)+r_2(1+\rho_1^2)]^3[r_1r_2(\rho_1\rho_2)^{3/2}(1-\hat{a}_2)(1+\hat{a}_2)^2]^{-1}$$

$$\times\left\{r_1^2(1-\rho_2^2)^2+r_2^2(1-\rho_1^2)^2+2r_1r_2[(\rho_1-\rho_2)^2+(1-\rho_1\rho_2)^2]\right\}^{-1} \quad . \tag{2.40}$$

Finally, from equations (2.34) and (2.40) the bivariate, saddlepoint approximation of the joint marginal density function of r_1 and r_2 (2.12) is

$$h(r_1,r_2) \sim \frac{(1-\rho_1^2)^{\frac{1}{2}}(1-\rho_2^2)^{\frac{1}{2}}(\frac{n}{2})(\frac{n}{2}-1)}{(\rho_1\rho_2)^{\frac{n}{2}} \qquad (r_1+r_2)^2}$$

$$\times\frac{1}{(2\pi i)^2}\iint a_2^{\frac{n}{2}-2}\frac{(1-a_1+a_2)^{\frac{1}{2}}(1+a_1+a_2)^{\frac{1}{2}}(1-a_2)}{[a_2(\beta_{11}-a_2)-a_2^2(\beta_{12}-a_1)(\beta_{21}-a_1)]}\left(\frac{\partial a_2}{\partial v}\right)^2\Bigg|_{v=0} dSdT$$

or

$$h(r_1,r_2) \sim \frac{(1-\rho_1^2)^{\frac{1}{2}}(1-\rho_2^2)^{\frac{1}{2}}\frac{n}{2}(\frac{n}{2}-1)}{(\rho_1\rho_2)^{\frac{n}{2}} \quad (r_1+r_2)^2} \cdot \frac{\hat{a}_2^{\frac{n}{2}-1}}{2\pi(\frac{n}{2}-2)}$$

$$\times 8\hat{a}_2^{\frac{1}{2}}(1-\hat{a}_2)^3(1+\hat{a}_2)(\rho_1\rho_2)^{3/2}\ r_1r_2(r_1+r_2)^2$$

$$\times\left\{(1-\hat{a}_2\rho_1\rho_2)[r_1(1+\rho_2^2)+r_2(1+\rho_1^2)][r_1(1-\rho_2^2)+r_2(1-\rho_1^2)]\right\}^{-1}$$

$$\times\left\{r_1^2(1-\rho_2^2)^2+r_2^2(1-\rho_1^2)^2 + 2r_1r_2[(\rho_1-\rho_2)^2+(1-\rho_1\rho_2)^2]\right\}^{-\frac{1}{2}} ,$$

and after cancellation

$$h(r_1,r_2) \sim \frac{2n(n-2)}{\pi(n-4)} \cdot \frac{(1-\rho_1^2)^{\frac{1}{2}}(1-\rho_2^2)^{\frac{1}{2}}(1-\hat{a}_2)^3(1+\hat{a}_2)r_1r_2}{(1-\hat{a}_2\rho_1\rho_2)}$$

$$\times\left[\frac{\hat{a}_2}{\rho_1\rho_2}\right]^{\frac{n}{2}-\frac{3}{2}}[r_1(1+\rho_2^2)+r_2(1+\rho_1^2)]^{-1}[r_1(1-\rho_2^2)+r_2(1-\rho_1^2)]^{-1}$$

$$\times\left\{r_1^2(1-\rho_2^2)^2+r_2^2(1-\rho_1^2)^2 + 2r_1r_2[(\rho_1-\rho_2)^2+(1-\rho_1\rho_2)^2]\right\}^{-\frac{1}{2}}, \tag{2.41}$$

where the relative error of the approximation is $O(n^{-1})$.

The product-moment correlation r is related to r_1 and r_2 by $r^2 = r_1r_2$. The approximate density function of r can be obtained from (2.41) as a marginal density. We do this in two stages. First let

$$\zeta = r_1r_2 \quad \text{and} \quad \xi = \frac{r_1}{r_2} \quad .$$

Then

$$r_1^2 = \zeta\xi \ , \quad r_2^2 = \frac{\zeta}{\xi} \quad \text{and} \quad \frac{\partial(r_1,r_2)}{\partial(\zeta,\xi)} = \frac{1}{2\xi} \quad .$$

We may also write

$$[r_1(1-\rho_2^2)+r_2(1-\rho_1^2)] = \sqrt{\zeta}\ [\sqrt{\xi}(1-\rho_2^2)+ \frac{1}{\sqrt{\xi}}\ (1-\rho_1^2)] \quad ,$$

$$[r_1(1+\rho_2^2)+r_2(1+\rho_1^2)] = \sqrt{\zeta}\ [\sqrt{\xi}(1+\rho_2^2)+ \frac{1}{\sqrt{\xi}}\ (1+\rho_1^2)]$$

and

$$\left\{r_1^2(1-\rho_2^2)^2+r_2^2(1-\rho_1^2)^2 + 2r_1r_2[(\rho_1-\rho_2)^2+(1-\rho_1\rho_2)^2]\right\}^{\frac{1}{2}}$$

$$= \sqrt{\zeta}\left\{ \xi(1-\rho_2^2)^2+\frac{1}{\xi}(1-\rho_1^2)^2 + 2[(\rho_1-\rho_2)^2+(1-\rho_1\rho_2)^2]\right\}^{\frac{1}{2}} \quad .$$

Thus the approximate joint density of ζ and ξ is

$$g(\zeta,\xi) \sim \frac{n(n-2)(1-\rho_1^2)^{\frac{1}{2}}(1-\rho_2^2)^{\frac{1}{2}}(1-\hat{a}_2)^3(1+\hat{a}_2)}{\pi(n-4)\ (1-\hat{a}_2\rho_1\rho_2)\quad \zeta^{\frac{1}{2}}\xi}$$

$$\times\left[\frac{a_2}{\rho_1\rho_2}\right]^{\frac{n}{2}-\frac{3}{2}} [\sqrt{\xi}(1+\rho_2^2) + \frac{1}{\sqrt{\xi}}(1+\rho_1^2)]^{-1}[\sqrt{\xi}(1-\rho_2^2)+\frac{1}{\sqrt{\xi}}(1-\rho_1^2)]^{-1}$$

$$\times\left\{\xi(1-\rho_2^2)^2 + \frac{1}{\xi}(1-\rho_1^2)^2 + 2[(\rho_1-\rho_2)^2+(1-\rho_1\rho_2)^2]\right\}^{-\frac{1}{2}}, \quad (2.42)$$

where $\hat{a}_2$ is now a function of ζ and ξ .

Then the approximate marginal density function of ζ is

$$f(\zeta) = \int_\xi g(\zeta,\xi)d\xi \sim \int [\chi(\zeta,\xi)]^m \quad \theta(\zeta,\xi)d\xi, \qquad (2.43)$$

where from (2.42)

$$\left.\begin{aligned} &\chi(\zeta,\xi) = \frac{\hat{a}_2}{\rho_1\rho_2}, \quad m = \tfrac{1}{2}(n-3) \\ \text{and}\quad &\theta(\zeta,\xi) = \frac{n(n-2)(1-\rho_1^2)^{\frac{1}{2}}(1-\rho_2^2)^{\frac{1}{2}}(1-\hat{a}_2)^3(1+\hat{a}_2)}{\pi(n-4)\ (1-\hat{a}_2\rho_1\rho_2)\ \zeta^{\frac{1}{2}}\ \xi} \\ &\times[\sqrt{\xi}(1+\rho_2^2)+ \frac{1}{\sqrt{\xi}}\ (1+\rho_1^2)]^{-1}[\sqrt{\xi}(1-\rho_2^2)+ \frac{1}{\sqrt{\xi}}(1-\rho_1^2)]^{-1} \\ &\times\left\{\xi(1-\rho_2^2)^2+ \frac{1}{\xi}(1-\rho_1^2)^2 + 2[(\rho_1-\rho_2)^2+(1-\rho_1\rho_2)^2]\right\}^{-\frac{1}{2}}. \end{aligned}\right\} (2.44)$$

Following Daniels [4], the saddle-point approximation of (2.43) is

$$f(\zeta) \sim \left[\frac{-2\pi\ \chi(\zeta,\hat{\xi})}{m\ \chi''(\zeta,\hat{\xi})}\right]^{\frac{1}{2}} \theta(\zeta,\hat{\xi})\ [\chi(\zeta,\hat{\xi})]^m \quad , \qquad (2.45)$$

where $\hat{\xi}$ is the solution of $\chi'(\zeta,\xi) = \frac{\partial\chi(\zeta,\xi)}{\partial\xi} = 0$ and $\chi''(\zeta,\hat{\xi}) = \left.\frac{\partial^2\chi(\zeta,\xi)}{\partial\xi^2}\right|_{\xi=\hat{\xi}}$.

From (2.44) and (2.26) we have

$$\chi(\zeta,\xi) = \frac{\hat{a}_2}{\rho_1\rho_2} = \frac{1}{\rho_1\rho_2}\,[1 + V - (V^2 + 2V)^{\frac{1}{2}}]$$

$$= \frac{1}{\rho_1\rho_2} \cdot \frac{(1+V)^2 - (V^2-2V)}{1 + V + (V^2+2V)^{\frac{1}{2}}}$$

$$= \frac{1}{\rho_1\rho_2}\,[1 + V + (V^2+2V)^{\frac{1}{2}}]^{-1} \quad , \qquad (2.46)$$

where

$$V = \frac{\left\{r_1(1-\rho_2^2)^2+r_2^2(1-\rho_1^2)^2+2r_1r_2[(\rho_1-\rho_2)^2+(1-\rho_1\rho_2)^2]\right\}}{8\, r_1r_2\rho_1\rho_2(1-r_1r_2)} \quad .$$

That is, in terms of ζ and ξ ,

$$V = \frac{\left\{\xi(1-\rho_2^2)^2+\frac{1}{\xi}(1-\rho_1^2)^2+2[(\rho_1-\rho_2)^2+(1-\rho_1\rho_2)^2]\right\}}{8\, \rho_1\rho_2(1-\zeta)} \quad . \qquad (2.47)$$

Thus, using (2.46) and (2.47),

$$\chi'(\zeta,\xi) = \frac{1}{\rho_1\rho_2}\,\frac{\partial\hat{a}_2}{\partial\xi} = \frac{1}{\rho_1\rho_2}\,\frac{d\hat{a}_2}{dV}\,\frac{\partial V}{\partial\xi}$$

$$= \frac{1}{\rho_1\rho_2}\left[1 - \frac{V + 1}{(V^2+2V)^{\frac{1}{2}}}\right]\frac{\left[(1-\rho_2^2)^2- \frac{1}{\xi^2}(1-\rho_1^2)^2\right]}{8\rho_1\rho_2(1-\zeta)} \quad ,$$

$$(2.48)$$

and since,

$$\left[1 - \frac{V + 1}{(V^2 + 2V)^{\frac{1}{2}}}\right] = -(V^2+2V)^{-\frac{1}{2}}[(V^2+2V)^{\frac{1}{2}}+V+1]^{-1}$$

has no finite zeros, $\chi'(\zeta,\xi) = 0$ if and only if $\frac{\partial V}{\partial\xi} = 0$, or

$$\hat{\xi} = \frac{1-\rho_1^2}{1-\rho_2^2} \quad . \qquad (2.49)$$

Then, using equation (2.49), (2.47) becomes

$$\hat{V} = \frac{[(1-\rho_1^2)(1-\rho_2^2)+(1-\rho_1^2)(1-\rho_2^2)+2(\rho_1-\rho_2)^2+2(1-\rho_1\rho_2)^2]}{8\rho_1\rho_2(1-\zeta)}$$

$$= \frac{2 - 4\rho_1\rho_2+2\rho_1^2\rho_2^2}{4\rho_1\rho_2(1-\zeta)}$$

$$= \frac{(1-\rho_1\rho_2)^2}{2\rho_1\rho_2(1-\zeta)} \quad , \qquad (2.50)$$

and so (2.46) becomes

$$\chi(\zeta,\hat{\xi}) = \frac{1}{\rho_1\rho_2}\left\{ 1 + \frac{(1-\rho_1\rho_2)^2}{2\rho_1\rho_2(1-\zeta)} + \frac{(1-\rho_1\rho_2)}{[2\rho_1\rho_2(1-\zeta)]^{\frac{1}{2}}} \times\left[\frac{(1-\rho_1\rho_2)^2+4\rho_1\rho_2(1-\zeta)}{2\rho_1\rho_2(1-\zeta)}\right]^{\frac{1}{2}} \right\}^{-1}$$

$$= 2(1-\zeta)\Big[2\rho_1\rho_2(1-\zeta) + (1-\rho_1\rho_2)^2 + (1-\rho_1\rho_2)[(1+\rho_1\rho_2)^2 - 4\rho_1\rho_2\zeta]^{\frac{1}{2}}\Big]^{-1}$$

$$= 2(1-\zeta)\Big[1+\rho_1^2\rho_2^2-2\rho_1\rho_2\zeta+(1-\rho_1\rho_2)[(1+\rho_1\rho_2)^2-4\rho_1\rho_2\zeta]^{\frac{1}{2}}\Big]^{-1} \quad . \qquad (2.51)$$

From (2.46), we obtain

$$\chi''(\zeta,\xi) = \frac{\partial^2\chi(\zeta,\xi)}{\partial\xi^2} = \frac{1}{\rho_1\rho_2}\left[\frac{d\hat{a}_2}{dV}\frac{\partial^2 V}{\partial\xi^2} + \frac{d^2\hat{a}_2}{dV^2}\left(\frac{\partial V}{\partial\xi}\right)^2\right] \quad .$$

But $\chi'(\zeta,\xi) = 0$ if and only if $\frac{\partial V}{\partial\xi} = 0$ and thus

$$\chi''(\zeta,\xi) = \frac{1}{\rho_1\rho_2} \cdot \frac{d\hat{a}_2}{dV} \left. \frac{\partial^2 V}{\partial \xi^2} \right|_{\xi=\hat{\xi}}$$

$$= \frac{1}{\rho_1\rho_2} \left[1 - \frac{\hat{V}+1}{(\hat{V}^2+2\hat{V})^{\frac{1}{2}}}\right] \frac{(1-\rho_1^2)^2}{4\rho_1\rho_2(1-\zeta)\hat{\xi}^3} ,$$

so that using (2.49),

$$\chi''(\zeta,\hat{\xi}) = \frac{(1-\rho_2^2)^3}{4\rho_1\rho_2(1-\zeta)(1-\rho_1^2)} \cdot \frac{(-1)}{\rho_1\rho_2[(\hat{V}^2+2\hat{V})^{\frac{1}{2}} + \hat{V} + 1]}$$

$$= \frac{(1-\rho_2^2)^3}{4\rho_1\rho_2(1-\zeta)(1-\rho_1^2)} \cdot \frac{-\chi(\zeta,\hat{\xi})}{(\hat{V}^2+2\hat{V})^{\frac{1}{2}}} .$$

Hence, by (2.50),

$$\frac{-\chi(\zeta,\hat{\xi})}{\chi''(\zeta,\hat{\xi})} = \frac{4\rho_1\rho_2(1-\rho_1^2)(1-\zeta)}{(1-\rho_2^2)^3} \cdot \frac{(1-\rho_1\rho_2)}{[2\rho_1\rho_2(1-\zeta)]^{\frac{1}{2}}} \times \left[\frac{(1+\rho_1\rho_2)^2 - 4\rho_1\rho_2\zeta}{2\rho_1\rho_2(1-\zeta)}\right]^{\frac{1}{2}}$$

$$= \frac{2(1-\rho_1^2)(1-\rho_1\rho_2)[(1+\rho_1\rho_2)^2 - 4\rho_1\rho_2\zeta]^{\frac{1}{2}}}{(1-\rho_2^2)^3} . \qquad (2.52)$$

From (2.44) and substituting $\hat{\xi}$ for ξ , we have

$$\theta(\zeta,\hat{\xi}) = \frac{n(n-2)(1-\rho_1^2)^{\frac{1}{2}}(1-\rho_2^2)^{\frac{1}{2}}}{\pi(n-4)} \; [1-\rho_1\rho_2\chi(\zeta,\hat{\xi})]^3$$

$$\times [1+\rho_1\rho_2\chi(\zeta,\hat{\xi})][1-\rho_1^2\rho_2^2\chi(\zeta,\hat{\xi})]^{-1} \; \zeta^{-\frac{1}{2}} \; \hat{\xi}^{-1}$$

$$\times [\sqrt{\hat{\xi}}(1+\rho_2^2) + \frac{1}{\sqrt{\hat{\xi}}}(1+\rho_1^2)]^{-1} \; [\sqrt{\hat{\xi}}(1-\rho_2^2) + \frac{1}{\sqrt{\hat{\xi}}}(1-\rho_1^2)]^{-1}$$

$$\times\left[\hat{\xi}(1-\rho_2^2)^2 + \frac{1}{\hat{\xi}}(1-\rho_1^2)^2 + 2[(\rho_1-\rho_2)^2 + (1-\rho_1\rho_2)^2]\right]^{-\frac{1}{2}} . \quad (2.53)$$

By (2.51) and (2.49) we write the following expressions in terms of ζ :

$$[1-\rho_1\rho_2\chi(\zeta,\hat{\xi})] = \left[1+\rho_1^2\rho_2^2-2\rho_1\rho_2\zeta+(1-\rho_1\rho_2)[(1+\rho_1\rho_2)^2-4\rho_1\rho_2\zeta]^{\frac{1}{2}}\right]^{-1}$$

$$\times\left[1+\rho_1^2\rho_2^2-2\rho_1\rho_2\zeta+(1-\rho_1\rho_2)[(1+\rho_1\rho_2)^2-4\rho_1\rho_2\zeta]^{\frac{1}{2}}\right.$$

$$\left.-2\rho_1\rho_2(1-\zeta)\right]$$

$$= \frac{(1-\rho_1\rho_2)\left[1-\rho_1\rho_2+[(1+\rho_1\rho_2)^2-4\rho_1\rho_2\zeta]^{\frac{1}{2}}\right]}{\left[1+\rho_1^2\rho_2^2-2\rho_1\rho_2\zeta+(1-\rho_1\rho_2)[(1+\rho_1\rho_2)^2-4\rho_1\rho_2\zeta]^{\frac{1}{2}}\right]} ,$$

$$[1+\rho_1\rho_2\chi(\zeta,\hat{\xi})] = \frac{[(1+\rho_1\rho_2)^2-4\rho_1\rho_2\zeta]^{\frac{1}{2}}\left[1-\rho_1\rho_2+[(1+\rho_1\rho_2)^2-4\rho_1\rho_2\zeta]^{\frac{1}{2}}\right]}{\left[1+\rho_1^2\rho_2^2-2\rho_1\rho_2\zeta+(1-\rho_1\rho_2)[(1+\rho_1\rho_2)^2-4\rho_1\rho_2\zeta]^{\frac{1}{2}}\right]} ,$$

$$(2.54)$$

$$[1-\rho_1^2\rho_2^2\chi(\zeta,\hat{\xi})] = \left[1+\rho_1^2\rho_2^2-2\rho_1\rho_2\zeta+(1-\rho_1\rho_2)[(1+\rho_1\rho_2)^2-4\rho_1\rho_2\zeta]^{\frac{1}{2}}\right]^{-1}$$

$$\times\left[1-\rho_1^2\rho_2^2-2\rho_1\rho_2\zeta(1-\rho_1\rho_2)\right.$$

$$\left.+(1-\rho_1\rho_2)[(1+\rho_1\rho_2)^2-4\rho_1\rho_2\zeta]^{\frac{1}{2}}\right]$$

$$= \frac{(1-\rho_1\rho_2)\left[1+\rho_1\rho_2(1-2\zeta)+[(1+\rho_1\rho_2)^2-4\rho_1\rho_2\zeta]^{\frac{1}{2}}\right]}{\left[1+\rho_1^2\rho_2^2-2\rho_1\rho_2\zeta+(1-\rho_1\rho_2)[(1+\rho_1\rho_2)^2-4\rho_1\rho_2\zeta]^{\frac{1}{2}}\right]} ,$$

$$\sqrt{\hat{\xi}}(1+\rho_2^2) + \frac{1}{\sqrt{\hat{\xi}}}(1+\rho_1^2) = \frac{1}{(1-\rho_1^2)^{\frac{1}{2}}(1-\rho_2^2)^{\frac{1}{2}}} \left[(1+\rho_2^2)(1-\rho_1^2)+(1+\rho_1^2)(1-\rho_2^2)\right]$$

$$= \frac{1+\rho_2^2-\rho_1^2-\rho_1^2\rho_2^2+1+\rho_1^2-\rho_2^2-\rho_1^2\rho_2^2}{(1-\rho_1^2)^{\frac{1}{2}} \quad (1-\rho_2^2)^{\frac{1}{2}}}$$

$$= \frac{2(1-\rho_1\rho_2)(1+\rho_1\rho_2)}{(1-\rho_1^2)^{\frac{1}{2}} \, (1-\rho_2^2)^{\frac{1}{2}}} \tag{2.55}$$

and

$$\sqrt{\hat{\xi}}(1-\rho_2^2) + \frac{1}{\sqrt{\hat{\xi}}}(1-\rho_1^2) = 2(1-\rho_1^2)^{\frac{1}{2}}(1-\rho_2^2)^{\frac{1}{2}} \quad .$$

Now (2.53) can be expressed as

$$\theta(\zeta,\hat{\xi}) = \frac{n(n-2)(1-\rho_1^2)^{\frac{1}{2}}(1-\rho_2^2)^{\frac{1}{2}}}{\pi(n-4) \quad \zeta^{\frac{1}{2}}} \cdot \frac{(1-\rho_2^2)}{(1-\rho_1^2)}$$

$$\times\Bigg[\left\{1+\rho_1^2\rho_2^2-2\rho_1\rho_2\zeta+(1-\rho_1\rho_2)[(1+\rho_1\rho_2)^2-4\rho_1\rho_2\zeta]^{\frac{1}{2}}\right\}^{-3}$$

$$\times(1-\rho_1\rho_2)^2\left\{1-\rho_1\rho_2+[(1+\rho_1\rho_2)^2-4\rho_1\rho_2\zeta]^{\frac{1}{2}}\right\}^4$$

$$\times\left\{1+\rho_1\rho_2(1-2\zeta)+[(1+\rho_1\rho_2)^2-4\rho_1\rho_2\zeta]^{\frac{1}{2}}\right\}^{-1}$$

$$\times[(1+\rho_1\rho_2)^2-4\rho_1\rho_2\zeta]^{\frac{1}{2}}\Bigg] \cdot \Big[4(1-\rho_1\rho_2)(1+\rho_1\rho_2)\Big]^{-1}$$

$$\times\left\{2(1-\rho_1^2)(1-\rho_2^2)+2[(\rho_1-\rho_2)^2+(1-\rho_1\rho_2)^2]\right\}^{-\frac{1}{2}} \quad ,$$

which may be simplified to

$$\theta(\zeta,\hat{\xi}) = \frac{n(n-2)(1-\rho_2^2)^{3/2}}{8\pi(n-4)(1-\rho_1^2)^{\frac{1}{2}}(1+\rho_1\rho_2)} \frac{[(1+\rho_1\rho_2)^2-4\rho_1\rho_2\zeta]^{\frac{1}{2}}}{\zeta^{\frac{1}{2}}}$$

$$\times\left[1-\rho_1\rho_2+[(1+\rho_1\rho_2)^2-4\rho_1\rho_2\zeta]^{\frac{1}{2}}\right]^4$$

$$\times\left[1+\rho_1^2\rho_2^2-2\rho_1\rho_2\zeta+(1-\rho_1\rho_2)[(1+\rho_1\rho_2)^2-4\rho_1\rho_2\zeta]^{\frac{1}{2}}\right]^{-3}$$

$$\times\left[1+\rho_1\rho_2(1-2\zeta)+[(1+\rho_1\rho_2)^2-4\rho_1\rho_2\zeta]^{\frac{1}{2}}\right]^{-1} \quad . \tag{2.56}$$

Then, substituting (2.52), (2.56) and (2.51) in (2.45), we obtain

$$f(\zeta) \sim \left[\frac{2\pi\cdot 2(1-\rho_1^2)(1-\rho_1\rho_2)[(1+\rho_1\rho_2)^2-4\rho_1\rho_2\zeta]^{\frac{1}{2}}}{\frac{1}{2}(n-3)\quad(1-\rho_2^2)^3}\right]^{\frac{1}{2}}$$

$$\times\ \frac{n(n-2)(1-\rho_2^2)^{3/2}\ [(1+\rho_1\rho_2)^2-4\rho_1\rho_2\zeta]^{\frac{1}{2}}}{8\pi(n-4)(1-\rho_1^2)^{\frac{1}{2}}\ (1+\rho_1\rho_2)\ \zeta^{\frac{1}{2}}}$$

$$\times\ \left[1-\rho_1\rho_2+[(1+\rho_1\rho_2)^2-4\rho_1\rho_2\zeta]^{\frac{1}{2}}\right]^4$$

$$\times\ \left[1+\rho_1^2\rho_2^2-2\rho_1\rho_2\zeta+(1-\rho_1\rho_2)[(1+\rho_1\rho_2)^2-4\rho_1\rho_2\zeta]^{\frac{1}{2}}\right]^{-3}$$

$$\times\ \left[1+\rho_1\rho_2(1-2\zeta)+[(1+\rho_1\rho_2)^2-4\rho_1\rho_2\zeta]^{\frac{1}{2}}\right]^{-1}$$

$$\times\ \frac{2^{\frac{1}{2}(n-3)}\quad(1-\zeta)^{\frac{1}{2}(n-3)}}{\left[1+\rho_1^2\rho_2^2-2\rho_1\rho_2\zeta+(1-\rho_1\rho_2)[(1+\rho_1\rho_2)^2-4\rho_1\rho_2\zeta]^{\frac{1}{2}}\right]^{\frac{1}{2}(n-3)}}\ ,$$

which, upon simplication, yields

$$f(\zeta) \sim \frac{2^{\frac{1}{2}(n-5)}\ n(n-2)\ (1-\rho_1\rho_2)^{\frac{1}{2}}\ (1-\zeta)^{\frac{1}{2}(n-3)}}{\sqrt{2\pi}\ (n-4)\ \sqrt{n-3}\ (1+\rho_1\rho_2)\quad \zeta^{\frac{1}{2}}}$$

$$\times[(1+\rho_1\rho_2)^2-4\rho_1\rho_2\zeta]^{3/4}\left[1-\rho_1\rho_2+[(1+\rho_1\rho_2)^2-4\rho_1\rho_2\zeta]^{\frac{1}{2}}\right]^4$$

$$\times\left[1+\rho_1^2\rho_2^2-2\rho_1\rho_2\zeta+(1-\rho_1\rho_2)[(1+\rho_1\rho_2)^2-4\rho_1\rho_2\zeta]^{\frac{1}{2}}\right]^{-\frac{1}{2}(n+3)}$$

$$\times\left[1+\rho_1\rho_2(1-2\zeta)+[(1+\rho_1\rho_2)^2-4\rho_1\rho_2\zeta]^{\frac{1}{2}}\right]^{-1}\ . \qquad (2.57)$$

Finally, letting $\zeta = r^2$, $d\zeta = 2\sqrt{\zeta}\ dr$, the approximate density function of the product-moment correlation r is

$$p(r) \sim \frac{2^{\frac{1}{2}(n-3)}\; n(n-2)(1-\rho_1\rho_2)^{\frac{1}{2}}(1-r^2)^{\frac{1}{2}(n-3)}}{\sqrt{2\pi}\;(n-4)\sqrt{n-3}\;(1+\rho_1\rho_2)}$$

$$\times[(1+\rho_1\rho_2)^2-4\rho_1\rho_2r^2]^{3/4}\left[1-\rho_1\rho_2+[(1+\rho_1\rho_2)^2-4\rho_1\rho_2r^2]^{\frac{1}{2}}\right]^4$$

$$\times\left[1+\rho_1^2\rho_2^2-2\rho_1\rho_2r^2+(1-\rho_1\rho_2)[(1+\rho_1\rho_2)^2-4\rho_1\rho_2r^2]^{\frac{1}{2}}\right]^{-\frac{1}{2}(n+3)}$$

$$\times\left[1+\rho_1\rho_2(1-2r^2)+[(1+\rho_1\rho_2)^2-4\rho_1\rho_2r^2]^{\frac{1}{2}}\right]^{-1} \quad . \tag{2.58}$$

But

$$1+\rho_1^2\rho_2^2-2\rho_1\rho_2r^2+(1-\rho_1\rho_2)[(1+\rho_1\rho_2)^2-4\rho_1\rho_2r^2]^{\frac{1}{2}}$$

$$= 1+\rho_1^2\rho_2^2+\tfrac{1}{2}[(1+\rho_1\rho_2)^2-4\rho_1\rho_2r^2]-\tfrac{1}{2}-\tfrac{1}{2}\rho_1^2\rho_2^2-\rho_1\rho_2$$

$$+(1-\rho_1\rho_2)[(1+\rho_1\rho_2)^2-4\rho_1\rho_2r^2]^{\frac{1}{2}}$$

$$= \tfrac{1}{2}\Big[(1-\rho_1\rho_2)^2+2(1-\rho_1\rho_2)[(1+\rho_1\rho_2)^2-4\rho_1\rho_2r^2]^{\frac{1}{2}}$$

$$+[(1+\rho_1\rho_2)^2-4\rho_1\rho_2r^2]\Big]$$

$$=\tfrac{1}{2}\left[1-\rho_1\rho_2+[(1+\rho_1\rho_2)^2-4\rho_1\rho_2r^2]^{\frac{1}{2}}\right]^2 \tag{2.59}$$

and

$$1+\rho_1\rho_2(1-2r^2)+[(1+\rho_1\rho_2)^2-4\rho_1\rho_2r^2]^{\frac{1}{2}}$$

$$= 1+\rho_1\rho_2+\tfrac{1}{2}[(1+\rho_1\rho_2)^2-4\rho_1\rho_2r^2]-\tfrac{1}{2}-\tfrac{1}{2}\rho_1^2\rho_2^2-\rho_1\rho_2$$

$$+[(1+\rho_1\rho_2)^2-4\rho_1\rho_2r^2]^{\frac{1}{2}}$$

$$= \tfrac{1}{2}\Big[(1-\rho_1\rho_2)(1+\rho_1\rho_2)+2\,[(1+\rho_1\rho_2)^2-4\rho_1\rho_2r^2]^{\frac{1}{2}}$$

$$+\,[(1+\rho_1\rho_2)^2-4\rho_1\rho_2r^2]\Big]$$

$$= \tfrac{1}{2}\left[[(1+\rho_1\rho_2)^2-4\rho_1\rho_2r^2]^{\frac{1}{2}}+(1-\rho_1\rho_2)\right]$$

$$\times\left[[(1+\rho_1\rho_2)^2-4\rho_1\rho_2r^2]^{\frac{1}{2}}+(1+\rho_1\rho_2)\right] \quad , \tag{2.60}$$

hence we may write $p(r)$ as

$$p(r) \sim \frac{2^{\frac{1}{2}(n-3)}\ n(n-2)\ (1-\rho_1\rho_2)^{\frac{1}{2}}\ (1-r^2)^{\frac{1}{2}(n-3)}}{\sqrt{2\pi}\ (n-4)\ \sqrt{n-3}\ \ (1+\rho_1\rho_2)}$$

$$\times\left[(1+\rho_1\rho_2)^2-4\rho_1\rho_2r^2\right]^{3/4}\left[[(1+\rho_1\rho_2)^2-4\rho_1\rho_2r^2]^{\frac{1}{2}}+(1+\rho_1\rho_2)\right]^{-1}$$

$$\times\, 2^{\frac{1}{2}(n+5)}\left[[(1+\rho_1\rho_2)^2-4\rho_1\rho_2r^2]^{\frac{1}{2}}+(1-\rho_1\rho_2)\right]^{-n} .$$

Since the approximate distribution depends only on the product of the serial correlations, let $\alpha = \rho_1\rho_2$, then

$$p(r) \sim \frac{2^{n+1}\ n(n-2)\ (1-\alpha)^{\frac{1}{2}}\ (1-r^2)^{\frac{1}{2}(n-3)}\ \left[(1+\alpha)^2-4\alpha r^2\right]^{3/4}}{\sqrt{2\pi}\ (n-4)\ \sqrt{n-3}\ (1+\alpha)}$$

$$\times\left[\ [(1+\alpha)^2-4\alpha r^2]^{\frac{1}{2}}+(1+\alpha)\right]^{-1}\left[\ [(1+\alpha)^2-4\alpha r^2]^{\frac{1}{2}}+(1-\alpha)\right]^{-n} , \quad (2.61)$$

where the error of the approximation is relatively $O(n^{-1})$. We note that the right member of (2.61) is an even function of r and hence, to the order considered, the mean and all of the odd moments are zero.

Having derived the approximate density function, $p(r)$ (2.61), we can verify Bartlett's [2] approximation of the variance of r, which is

$$\mathrm{var}(r) \sim \frac{1}{n}\left[\frac{1+\rho_1\rho_2}{1-\rho_1\rho_2}\right] .$$

Write

$$1-r^2 = 1 + \frac{1}{4\alpha}\left[(1+\alpha)^2-4\alpha r^2\right] - \frac{(1+\alpha)^2}{4\alpha}$$

$$= \frac{1}{4\alpha}\left\{\left[(1+\alpha)^2-4\alpha r^2\right] - (1-\alpha)^2\right\}$$

$$= \frac{1}{4\alpha}\left\{\left[(1+\alpha)^2-4\alpha r^2\right]^{\frac{1}{2}}-(1-\alpha)\right\}\left\{\left[(1+\alpha)^2-4\alpha r^2\right]^{\frac{1}{2}}+(1-\alpha)\right\} . \quad (2.62)$$

Then (2.61) becomes

$$p(r) \sim \frac{2^4\, n(n-2)\, (1-\alpha)^{\frac{1}{2}}\, [(1+\alpha)^2-4\alpha r^2]^{3/4}}{\sqrt{2\pi}\, (n-4)\sqrt{n-3}\, (1+\alpha)\, \alpha^{\frac{1}{2}(n-3)}}$$

$$\times \left[\frac{[(1+\alpha)^2-4\alpha r^2]^{\frac{1}{2}} - (1-\alpha)}{[(1+\alpha)^2-4\alpha r^2]^{\frac{1}{2}} + (1-\alpha)}\right]^{\frac{1}{2}(n-3)}$$

$$\times \left[[(1+\alpha)^2-4\alpha r^2]^{\frac{1}{2}} + (1-\alpha)\right]^{-3}\left[[(1+\alpha)^2-4\alpha r^2]^{\frac{1}{2}} + (1+\alpha)\right]^{-1},$$

and hence,

$$\operatorname{var}(r) \sim \frac{2^4\, n(n-2)\, (1-\alpha)^{\frac{1}{2}}}{\sqrt{2\pi}\, (n-4)\sqrt{n-3}\, (1+\alpha)\, \alpha^{\frac{1}{2}(n-3)}}$$

$$\times \int_{-1}^{1} \left[\frac{[(1+\alpha)^2-4\alpha r^2]^{\frac{1}{2}} - (1-\alpha)}{[(1+\alpha)^2-4\alpha r^2]^{\frac{1}{2}} + (1-\alpha)}\right]^{\frac{1}{2}(n-3)} \cdot \left[(1+\alpha)^2-4\alpha r^2\right]^{3/4}$$

$$\times \left[[(1+\alpha)^2-4\alpha r^2]^{\frac{1}{2}} + (1-\alpha)\right]^{-3}$$

$$\times \left[[(1+\alpha)^2-4\alpha r^2]^{\frac{1}{2}} + (1+\alpha)\right]^{-1} r^2\, dr\,. \qquad (2.63)$$

Expanding $\ln \left[\frac{[(1+\alpha)^2-4\alpha r^2]^{\frac{1}{2}} - (1-\alpha)}{[(1+\alpha)^2-4\alpha r^2]^{\frac{1}{2}} + (1-\alpha)}\right]$ in Maclaurin's series, we have

$$\ln \left[\frac{[(1+\alpha)^2-4\alpha r^2]^{\frac{1}{2}} - (1-\alpha)}{[(1+\alpha)^2-4\alpha r^2]^{\frac{1}{2}} + (1-\alpha)}\right] = \ln \alpha - \left(\frac{1-\alpha}{1+\alpha}\right) r^2 + \dots \,. \qquad (2.64)$$

Now using the first two terms of the series as an approximation, (2.63) can be written as

$$\operatorname{var}(r) \sim \frac{2^4\ n(n-2)\ (1-\alpha)^{\frac{1}{2}}}{\sqrt{2\pi}\ (n-4)\ \sqrt{n-3}\ (1+\alpha)\alpha^{\frac{1}{2}(n-3)}}$$

$$\times \int_{-1}^{1} \exp\left\{\tfrac{1}{2}(n-3)\left[\ell n\alpha - \frac{1-\alpha}{1+\alpha}\ r^2\right]\right\}\left[(1+\alpha)^2-4\alpha r^2\right]^{3/4}$$

$$\times\left[[(1+\alpha)^2-4\alpha r^2]^{\frac{1}{2}} + (1-\alpha)\right]^{-3}$$

$$\times\left[[(1+\alpha)^2-4\alpha r^2]^{\frac{1}{2}} + (1+\alpha)\right]^{-1} r^2\ dr\ , \qquad (2.65)$$

from which, using Daniels' asymptotic expansion (see (3.1) in [3]), we get

$$\operatorname{var}(r) \sim \frac{2^4\ n(n-2)\ (1-\alpha)^{\frac{1}{2}}}{\sqrt{2n}\ (n-4)\ \sqrt{n-3}\ (1+\alpha)\ \alpha^{\frac{1}{2}(n-3)}}$$

$$\times\ \alpha^{\frac{1}{2}(n-3)}\left[\frac{2\pi}{(n-3)\ \frac{1-\alpha}{1+\alpha}}\right]^{\frac{1}{2}}\left[\frac{1}{2(n-3)\ \frac{1-\alpha}{1+\alpha}}\right]\ \frac{(1+\alpha)^{3/2}}{2^3(1+\alpha)}\ .$$

Upon simplification this becomes

$$\operatorname{var}(r) \sim \frac{n(n-2)\quad (1+\alpha)}{(n-4)(n-3)^2(1-\alpha)} \sim \frac{1}{n}\left[\frac{1+\rho_1\rho_2}{1-\rho_1\rho_2}\right]\ . \qquad (2.66)$$

Thus the variance of r, that is $\mathcal{E}(r^2)$, is $O(\frac{1}{n})$.

By an adjustment of the exponents of the terms on the right of (2.61), we obtain a form which is readily renormalized by using the result of Appendix IV. Write (2.61) as

$$p(r) \sim K(1-r^2)^{\frac{1}{2}(n-3)}\left[(1+\alpha)^2-4\alpha r^2\right]^{3/4}\left[[(1+\alpha)^2-4\alpha r^2]^{\frac{1}{2}}+(1+\alpha)\right]^{-1}$$

$$\times\left[[(1+\alpha)^2-4\alpha r^2]^{\frac{1}{2}}+(1-\alpha)\right]^{-n}\ , \qquad (2.67)$$

where $K = \dfrac{2^{n+1}\ n(n-2)(1-\alpha)^{\frac{1}{2}}}{\sqrt{2\pi}\ (n-4)\ \sqrt{n-3}\ (1+\alpha)}$.

Equation (2.67) may be expressed as

$$p(r) \sim K \frac{(1-r^2)^{\frac{1}{2}(n-3)}\left[(1+\alpha)^2-4\alpha r^2\right]^{5/4}}{\left[\left[(1+\alpha)^2-4\alpha r^2\right]^{\frac{1}{2}}+(1+\alpha)\right]^{3/2}}$$

$$\times \frac{\left[\left[(1+\alpha)^2-4\alpha r^2\right]^{\frac{1}{2}}+(1-\alpha)\right]^{\delta}}{\left[\left[(1+\alpha)^2-4\alpha r^2\right]^{\frac{1}{2}}+(1-\alpha)\right]^{n+\delta}}$$

$$\times \frac{\left[\left[(1+\alpha)^2-4\alpha r^2\right]^{\frac{1}{2}}+(1+\alpha)\right]^{\frac{1}{2}}}{\left[(1+\alpha)^2-4\alpha r^2\right]^{\frac{1}{2}}} . \qquad (2.68)$$

To determine the most convenient δ, expand the following functions of r^2 as Maclaurin's series:

for

$$g_1(r^2) = \left[(1+\alpha)^2-4\alpha r^2\right]^{5/4} ,$$

$$g_1'(r^2) = -5\alpha\left[(1+\alpha)^2-4\alpha r^2\right]^{1/4} ,$$

$$g_1(0) = (1+\alpha)^{5/2} ,$$

$$g_1'(0) = -5\alpha(1+\alpha)^{\frac{1}{2}}$$

giving

$$g_1(r^2) = (1+\alpha)^{5/2} - 5\alpha(1+\alpha)^{\frac{1}{2}}r^2 + O(r^4)$$

$$= (1+\alpha)^{5/2}\left[1 - \frac{5\alpha}{(1+\alpha)^2} r^2\right] + O(r^4)$$

$$= (1+\alpha)^{5/2}\left[1 - r^2\right]^{\frac{5\alpha}{(1+\alpha)^2}} + O(r^4) ; \qquad (2.69)$$

for

$$g_2(r^2) = \left[\left[(1+\alpha)^2-4\alpha r^2\right]^{\frac{1}{2}} + (1+\alpha)\right]^{-3/2}$$

$$g_2'(r^2) = 3\alpha\left[(1+\alpha)^2-4\alpha r^2\right]^{-\frac{1}{2}}\left[\left[(1+\alpha)^2-4\alpha r^2\right]^{\frac{1}{2}} + (1+\alpha)\right]^{-5/2} ,$$

$$g_2(0) = \left[2(1+\alpha)\right]^{-3/2} ,$$

$$g_2'(0) = 3\alpha(2)^{-5/2}(1+\alpha)^{-7/2}$$

giving

$$g_2(r^2) = [2(1+\alpha)]^{-3/2}+3\alpha 2^{-5/2}(1+\alpha)^{-7/2}r^2+O(r^4)$$

$$= [2(1+\alpha)]^{-3/2}\left[1+\frac{3\alpha}{2(1+\alpha)^2}\; r^2\right] + O(r^4)$$

$$= [2(1+\alpha)]^{-3/2}\left[1-r^2\right]^{\frac{-3\alpha}{2(1+\alpha)^2}} + O(r^4)\ ; \qquad (2.70)$$

and for

$$g_3(r^2) = \left[[(1+\alpha)^2-4\alpha r^2]^{\frac{1}{2}} + (1-\alpha)\right]^{\delta}$$

$$g_3'(r^2) = -2\alpha\delta\,[(1+\alpha)^2-4\alpha r^2]^{-\frac{1}{2}}$$

$$\times\left[[(1+\alpha)^2-4\alpha r^2]^{\frac{1}{2}}+(1-\alpha)\right]^{\delta-1} \ ,$$

$$g_3(0) = 2^{\delta}\ ,$$

$$g_3'(0) = -2^{\delta}\alpha\delta(1+\alpha)^{-1}$$

giving

$$g_3(r^2) = 2^{\delta}-2^{\delta}\alpha\delta(1+\alpha)^{-1}r^2 + O(r^4)$$

$$= 2^{\delta}\left[1-\frac{\alpha\delta}{(1+\alpha)}\, r^2\right] + O(r^4)$$

$$= 2^{\delta}\left[1-r^2\right]^{\frac{\alpha\delta}{(1+\alpha)}} + O(r^4)\ . \qquad (2.71)$$

Since the variance of r is of $O(n^{-1})$, see (2.66), we may take r^2 to be $O(n^{-1})$ over the effective range of r. Hence, omitting the $O(r^4)$ terms in (2.69), (2.70) and (2.71), substituting them in (2.68) involves an error which is relatively $O(n^{-2})$. Thus, we have

$$p(r) \sim 2^{\delta-3/2}(1+\alpha)\, K\left[1-r^2\right]^{\frac{1}{2}(n-3)+\frac{5\alpha}{(1+\alpha)^2}-\frac{3\alpha}{2(1+\alpha)^2}+\frac{\alpha\delta}{(1+\alpha)}}$$

$$\times\left[[(1+\alpha)^2-4\alpha r^2]^{\frac{1}{2}} + (1-\alpha)\right]^{-(n+\delta)}$$

$$\times\frac{\left[[(1+\alpha)^2-4\alpha r^2]^{\frac{1}{2}} + (1+\alpha)\right]^{\frac{1}{2}}}{[(1+\alpha)^2-4\alpha r^2]^{\frac{1}{2}}}\ ,$$

which may be written as

$$p(r) \sim K' \frac{\left[1-r^2\right]^{\frac{1}{2}\left(n-3+\frac{7\alpha}{(1+\alpha)^2}+\frac{2\alpha\delta}{(1+\alpha)}\right)}}{\left[[(1+\alpha)^2-4\alpha r^2]^{\frac{1}{2}}+(1+\alpha)\right]^{n+\delta}}$$

$$\times \frac{\left[[(1+\alpha)^2-4\alpha r^2]^{\frac{1}{2}}+(1+\alpha)\right]^{\frac{1}{2}}}{\left[(1+\alpha)^2-4\alpha r^2\right]^{\frac{1}{2}}} , \qquad (2.72)$$

where $K' = 2^{\delta-\frac{3}{2}}(1+\alpha)K$. Now we choose δ such that

$$n+\delta = n-3+\frac{7\alpha}{(1+\alpha)^2}+\frac{2\alpha\delta}{(1+\alpha)}+\frac{3}{2}$$

or

$$\delta\left[1-\frac{2\alpha}{(1+\alpha)}\right] = \frac{7\alpha}{(1+\alpha)^2}-\frac{3}{2} .$$

Hence

$$\delta\left[\frac{1-\alpha}{1+\alpha}\right] = -\frac{[3\alpha^2-8\alpha+3]}{2(1+\alpha)^2} ,$$

and thus,

$$\delta = \frac{-[3\alpha^2-8\alpha+3]}{2(1+\alpha)(1-\alpha)} .$$

Also let

$$\begin{aligned} N &= n+\frac{7\alpha}{(1+\alpha)^2}+\frac{2\alpha\delta}{(1+\alpha)} \\ &= n+\frac{7\alpha}{(1+\alpha)^2}-\frac{[3\alpha^3-8\alpha^2+3\alpha]}{(1+\alpha)^2(1-\alpha)} \\ &= n+\frac{\alpha[4+\alpha-3\alpha^2]}{(1+\alpha)^2(1-\alpha)} \\ &= n+\frac{\alpha(4-3\alpha)}{(1-\alpha^2)} . \qquad (2.73) \end{aligned}$$

Then (2.72) may be written as

$$p(r) \sim K' \frac{\left[1-r^2\right]^{\frac{1}{2}(N-3)}}{\left[\left[(1+\alpha)^2-4\alpha r^2\right]^{\frac{1}{2}} + (1-\alpha)\right]^{N-3/2}}$$

$$\times \frac{\left[\left[(1+\alpha)^2-4\alpha r^2\right]^{\frac{1}{2}} + (1+\alpha)\right]^{\frac{1}{2}}}{\left[(1+\alpha)^2 - 4\alpha r^2\right]^{\frac{1}{2}}} \quad . \tag{2.74}$$

To normalize $p(r)$, we must evaluate the integral

$$\int_{-1}^{1} p(r)\, dr = 2 \int_{0}^{1} p(r)\, dr \; .$$

Using (2.74),

$$\int_{0}^{1} p(r)\, dr \sim K' \int_{0}^{1} \frac{(1-r^2)^{\frac{1}{2}(N-3)}}{\left[\left[(1+\alpha)^2-4\alpha r^2\right]^{\frac{1}{2}} + (1-\alpha)\right]^{N-3/2}}$$

$$\times \frac{\left[\left[(1+\alpha)^2-4\alpha r^2\right]^{\frac{1}{2}} + (1+\alpha)\right]^{\frac{1}{2}}}{\left[(1+\alpha)^2 - 4\alpha r^2\right]^{\frac{1}{2}}}\, dr.$$

From Appendix IV we see that

$$\int_{0}^{1} p(r)\, dr \sim K'\, J\left[\tfrac{1}{2}(N-3),\, N-\tfrac{3}{2},\, \alpha\right] \quad ,$$

and since $N - \frac{3}{2} = 2[\frac{1}{2}(N-3)] + \frac{3}{2}$, we may use the relationship (IV.13), to give

$$\int_{0}^{1} p(r)\, dr \sim K'\, 2^{-(N-1)}\, B[\tfrac{1}{2}(N-1), \tfrac{1}{2}]\, (1-\alpha)^{-\frac{1}{2}} \quad . \tag{2.75}$$

Finally, using (2.74) and (2.75), the approximate density function of the product-moment correlation r in renormalized form is

$$p(r) \sim \frac{2^{N-2}(1-\alpha)^{\frac{1}{2}}}{B[\frac{1}{2}(N-1),\frac{1}{2}]} \cdot \frac{[1-r^2]^{\frac{1}{2}(N-3)}}{\left[[(1+\alpha)^2-4\alpha r^2]^{\frac{1}{2}}+(1-\alpha)\right]^{N-3/2}}$$

$$\times \frac{\left[[(1+\alpha)^2-4\alpha r^2]^{\frac{1}{2}}+(1+\alpha)\right]^{\frac{1}{2}}}{\left[(1+\alpha)^2-4\alpha r^2\right]^{\frac{1}{2}}}, \qquad (2.76)$$

where $N = n + \frac{\alpha(4-3\alpha)}{(1-\alpha^2)}$, $\alpha = \rho_1\rho_2$ and the error of the approximation is relatively $O(n^{-1})$. Note that (2.76) depends on the auto-correlations ρ_1 and ρ_2 only through the product $\alpha = \rho_1\rho_2$ and that, for $\alpha = 0$, the approximate distribution reduces to the familiar null distribution of the product-moment correlation with known means, which is

$$p(r) \sim \frac{(1-r^2)^{\frac{1}{2}(n-3)}}{B[\frac{1}{2}(n-1),\frac{1}{2}]} .$$

CHAPTER III

THE APPROXIMATE DISTRIBUTION OF THE CORRELATION BETWEEN TWO STATIONARY, LINEAR MARKOV SERIES WITH FITTED MEANS

In this chapter we derive the approximate null-distribution of the sample product-moment correlation, r, when the series $\{x_i\}$ and $\{y_i\}$ are each of the linear, stationary Markov type where the means are fitted and the auto-correlations are ρ_1 and ρ_2, respectively. In this derivation, which follows the general stages of the procedure in Chapter II, we encounter the matrix $\underline{B}$ (3.6) whose determinant, $|\underline{B}|$, is evaluated approximately in Appendix III.

Consider the two stationary, linear Markov series

$$\left.\begin{aligned} & x_i = \rho_1 x_{i-1} + \epsilon_i \quad \text{for all } i \\ \text{and} \\ & y_j = \rho_2 y_{j-1} + \eta_j \quad \text{for all } j , \end{aligned}\right\} \qquad (3.1)$$

where ϵ's and η's are independent $N(0,1)$ variables. We seek an approximate distribution of the product-moment correlation r, where

$$r^2 = r_1 r_2$$

and r_1 and r_2 are defined to be:

$$r_1 = \frac{\sum_{i=1}^{n} (x_i-\bar{x})(y_i-\bar{y})}{\sum_{i=1}^{n} (x_i-\bar{x})^2} \qquad (3.2)$$

and

$$r_2 = \frac{\sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n} (y_i - \bar{y})^2} , \tag{3.3}$$

and where

$$\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i \quad \text{and} \quad \bar{y} = \frac{1}{n} \sum_{i=1}^{n} y_i .$$

Since the case where the means are known was done in detail, we shall refer to and use freely the intermediate results and notations of Chapter II for the present case where the means are fitted. Of course, we must keep in mind that when using any of the Chapter II relationships r_1 and r_2 of Chapter II will be replaced by r_1 and r_2 as defined by (3.2) and (3.3), respectively. Let

$$C^* = \sum_{i=1}^{n} (x_i - \bar{x})^2 ,$$

$$D^* = \sum_{i=1}^{n} (y_i - \bar{y})^2$$

and

$$E^* = \sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y}) .$$

The joint moment-generating function for the distribution of C^*, D^* and E^* is

$$M^*(T,S,U) = \mathcal{E}\left[e^{TC^* + SD^* + UE^*}\right]$$

$$= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{TC^* + SD^* + UE^*} \, dF(x_1, \ldots, x_n, y_1, \ldots, y_n) . \tag{3.4}$$

The joint distribution of $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n$, see (3.1), is

$$dF(x_1,\ldots,x_n,y_1,\ldots,y_n) = (2\pi)^{-n} |\Sigma_x|^{-\frac{1}{2}} |\Sigma_y|^{-\frac{1}{2}}$$

$$\times \exp\left\{-\tfrac{1}{2}\left[\underline{x}'\, \Sigma_x^{-1}\, \underline{x} + \underline{y}'\, \Sigma_y^{-1}\, \underline{y}\right]\right\}$$

$$\times\, dx_1 \ldots dx_n\, dy_1 \ldots dy_n \quad ,$$

where $\underline{x} = (x_1, x_2, \ldots, x_n)'$, $\underline{y} = (y_1, y_2, \ldots, y_n)'$ and the corresponding covariance matrices, Σ_x and Σ_y, may be found in Chapter II. Thus, using (2.2) and (2.3), (3.4) is

$$M^*(T,S,U) = \frac{(1-\rho_1^2)^{\frac{1}{2}}(1-\rho_2^2)^{\frac{1}{2}}}{(2\pi)^n}$$

$$\times \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp\left\{TC^* + SD^* + UE^* \;\; -\tfrac{1}{2}\left[\underline{x}'\, \Sigma_x^{-1}\, \underline{x} + \underline{y}'\, \Sigma_y^{-1}\, \underline{y}\right]\right\}$$

$$\times\, dx_1 \ldots dx_n\, dy_1 \ldots dy_n \quad . \tag{3.5}$$

Denote the exponent of the integrand by Q, that is,

$$Q = -\tfrac{1}{2}\left[\, \underline{x}'\, \Sigma_x^{-1}\, \underline{x} + \underline{y}'\, \Sigma_y^{-1}\, \underline{y} - 2TC^* - 2SD^* - 2UE^* \right] .$$

We know that

$$\underline{x}'\, \Sigma_x^{-1}\, \underline{x} = x_1^2 + (1+\rho_1^2) \sum_{i=2}^{n-1} x_i^2 - 2\rho_1 \sum_{i=1}^{n-1} x_i\, x_{i+1} + x_n^2$$

and

$$\underline{y}' \, \underline{V}_y^{-1} \, \underline{y} = y_1^2 + (1+\rho_2^2) \sum_{i=2}^{n-1} y_i^2 - 2\rho_2 \sum_{i=1}^{n-1} y_i \, y_{i+1} + y_n^2$$

Now

$$-2TC^* = -2T\left[\sum_{i=1}^{n} x_i^2 - \frac{1}{n} \sum_{i=1}^{n} x_i \sum_{i=1}^{n} x_i\right]$$

$$= -2T\Big[(x_1^2 + \ldots + x_n^2) - \frac{1}{n}(x_1^2 + \ldots + x_n^2)$$

$$- \frac{2}{n} (x_1x_2+\ldots+x_{n-1}x_n+x_1x_3+\ldots+x_{n-2}x_n$$

$$+\ldots+x_1x_{n-1}+x_2x_n+x_1x_n)\Big]$$

$$= -2T\Big[(1 - \frac{1}{n})x_1^2 + (1 - \frac{1}{n})x_n^2 + (1 - \frac{1}{n}) \sum_{i=2}^{n-1} x_i^2$$

$$- \frac{2}{n}\Big(\sum_{i=1}^{n-1} x_ix_{i+1} + \sum_{i=1}^{n-2} x_ix_{i+2} + \ldots + \sum_{i=1}^{2} x_ix_{i+n-2}$$

$$+ x_1x_n\Big)\Big] \quad .$$

Similarly,

$$-2SD^* = -2S\left[\sum_{i=1}^{n} y_i^2 - \frac{1}{n} \sum_{i=1}^{n} y_i \sum_{i=1}^{n} y_i\right]$$

$$= -2S\Big[(1 - \frac{1}{n})y_1^2 + (1 - \frac{1}{n})y_n^2 + (1 - \frac{1}{n}) \sum_{i=2}^{n-1} y_i^2$$

$$- \frac{2}{n}\Big(\sum_{i=1}^{n-1} y_i \, y_{i+1} + \sum_{i=1}^{n-2} y_i \, y_{i+2} + \ldots + \sum_{i=1}^{2} y_i \, y_{i+n-2}$$

$$+ y_1y_n\Big)\Big]$$

and

$$-2UE^* = -2U\left[\sum_{i=1}^{n} x_i\, y_i - \frac{1}{n}\sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i\right]$$

$$= -2U\Big[(x_1y_1+\ldots+x_ny_n) - \frac{1}{n}(x_1y_1+\ldots+x_ny_n)$$

$$- \frac{1}{n}(x_1y_2+\ldots+x_1y_n+x_2y_1+x_2y_3+\ldots+x_2y_n$$

$$+\ldots+x_ny_1+\ldots+x_ny_{n-1})\Big]$$

$$= -2U\Big[(1-\frac{1}{n})\sum_{i=1}^{n} x_i\, y_i - \frac{1}{n}\sum_{i=1}^{n-1} x_i\, y_{i+1} - \frac{1}{n}\sum_{i=2}^{n} x_i\, y_{i-1}$$

$$- \frac{1}{n}\sum_{i=1}^{n-2} x_i\, y_{i+2} - \frac{1}{n}\sum_{i=3}^{n} x_i\, y_{i-2}$$

$$- \ldots - \frac{1}{n}\, x_1\, y_n - \frac{1}{n}\, x_ny_1\Big] \;.$$

Then Q is

$$Q = -\tfrac{1}{2}\Big\{[(1 - 2T(1-\frac{1}{n})]x_1^2 + [1 - 2T(1-\frac{1}{n})]x_n^2$$

$$+ [1 + \rho_1^2 - 2T(1-\frac{1}{n})]\sum_{i=2}^{n-1} x_i^2 + [-2\rho_1 + \frac{4T}{n}]\sum_{i=1}^{n-1} x_i\, x_{i+1}$$

$$+ \frac{4T}{n}\sum_{i=1}^{n-2} x_i\, x_{i+2} + \ldots + \frac{4T}{n}\sum_{i=1}^{2} x_i\, x_{i+n-2} + \frac{4T}{n}\, x_1\, x_n$$

$$+ [1 - 2S(1-\frac{1}{n})]y_1^2 + [1 - 2S(1-\frac{1}{n})]y_n^2$$

$$+ [1 + \rho_2^2 - 2S(1-\frac{1}{n})]\sum_{i=2}^{n-1} y_i^2 + [-2\rho_2 + \frac{4S}{n}]\sum_{i=1}^{n-1} y_i\, y_{i+1}$$

continued

$$+ \frac{4S}{n} \sum_{i=1}^{n-2} y_i\, y_{i+2} + \cdots + \frac{4S}{n} \sum_{i=1}^{2} y_i\, y_{i+n-2} + \frac{4S}{n}\, y_1 y_n$$

$$- 2U\left(1 - \frac{1}{n}\right) \sum_{i=1}^{n} x_i\, y_i + \frac{2U}{n} \sum_{i=1}^{n-1} x_i\, y_{i+1}$$

$$+ \frac{2U}{n} \sum_{i=2}^{n} x_i\, y_{i-1} + \frac{2U}{n} \sum_{i=1}^{n-2} x_i\, y_{i+2} + \frac{2U}{n} \sum_{i=3}^{n} x_i\, y_{i-2}$$

$$+ \cdots + \frac{2U}{n}\, x_1 y_n + \frac{2U}{n}\, x_n y_1 \Bigg\}$$

$$= - \tfrac{1}{2}\ \underline{x}^{*\prime}\ \underline{B}\ \underline{x}^{*} \quad ,$$

where $\underline{x}^* = (x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n)'$ and $\underline{B}$ is the $2n \times 2n$ partitioned matrix

$$\underline{B} = \left[\begin{array}{c|c} \underline{R}_1 & \underline{R}_3 \\ \hline \underline{R}_3 & \underline{R}_2 \end{array}\right] , \qquad (3.6)$$

with $n \times n$ submatrices

$$\underline{R}_1 = \begin{bmatrix} a+\alpha & -\rho_1+\alpha & \alpha & \cdots & \alpha & \alpha \\ -\rho_1+\alpha & b+\alpha & -\rho_1+\alpha & \cdots & \alpha & \alpha \\ \alpha & -\rho_1+\alpha & b+\alpha & \cdots & \alpha & \alpha \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \alpha & \alpha & \alpha & \cdots & b+\alpha & -\rho_1+\alpha \\ \alpha & \alpha & \alpha & \cdots & -\rho_1+\alpha & a+\alpha \end{bmatrix} ,$$

$$\underline{R}_2 = \begin{bmatrix} c+\beta & -\rho_2+\beta & \beta & \cdots & \beta & \beta \\ -\rho_2+\beta & d+\beta & -\rho_2+\beta & \cdots & \beta & \beta \\ \beta & -\rho_2+\beta & d+\beta & \cdots & \beta & \beta \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \beta & \beta & \beta & \cdots & d+\beta & -\rho_2+\beta \\ \beta & \beta & \beta & \cdots & -\rho_2+\beta & c+\beta \end{bmatrix},$$

and

$$\underline{R}_3 = \begin{bmatrix} -U+\gamma & \gamma & \gamma & \cdots & \gamma & \gamma \\ \gamma & -U+\gamma & \gamma & \cdots & \gamma & \gamma \\ \gamma & \gamma & -U+\gamma & \cdots & \gamma & \gamma \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \gamma & \gamma & \gamma & \cdots & -U+\gamma & \gamma \\ \gamma & \gamma & \gamma & \cdots & \gamma & -U+\gamma \end{bmatrix},$$

where

$$a = 1 - 2T, \quad b = 1 + \rho_1^2 - 2T$$

$$c = 1 - 2S, \quad d = 1 + \rho_2^2 - 2S$$

$$\alpha = \frac{2T}{n}, \quad \beta = \frac{2S}{n} \quad \text{and} \quad \gamma = \frac{U}{n} .$$

Then (3.5) is

$$M^*(T,S,U) = \frac{(1-\rho_1^2)^{\frac{1}{2}}(1-\rho_2^2)^{\frac{1}{2}}}{(2\pi)^n} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp[-\tfrac{1}{2}\underline{x}^{*\prime}\,\underline{B}\,\underline{x}^*]dx_1 \ldots dx_n dy_1 \ldots dy_n$$

and hence

$$M^*(T,S,U) = \frac{(1-\rho_1^2)^{\frac{1}{2}}(1-\rho_2^2)^{\frac{1}{2}}}{|\underline{B}|^{\frac{1}{2}}} . \tag{3.7}$$

Using the result (III.51) of Appendix III, we have

$$|\underline{B}| \sim \frac{\rho_1^{n-1}\ \rho_2^{n-1}\ (1-\rho_1)^2(1-\rho_2)^2}{a_2^{n-1}}$$

$$\times \frac{[a_2(\beta_{11}-a_2)-a_2^2(\beta_{12}-a_1)(\beta_{21}-a_1)]^2}{(1-a_1+a_2)(1+a_1+a_2)^3(1-a_2)^2} ,$$

where the error is $O(\frac{1}{n})$.

Thus the joint moment-generating function is approximately

$$M^*(T,S,U) \sim \frac{(1-\rho_1^2)^{\frac{1}{2}}(1-\rho_2^2)^{\frac{1}{2}}}{(1-\rho_1)(1-\rho_2)} \cdot \frac{a_2^{\frac{1}{2}(n-1)}}{(\rho_1\rho_2)^{\frac{1}{2}(n-1)}}$$

$$\times \frac{(1-a_1+a_2)^{\frac{1}{2}}(1+a_1+a_2)^{3/2}(1-a_2)}{[a_2(\beta_{11}-a_2)-a_2^2(\beta_{12}-a_1)(\beta_{21}-a_1)]} , \tag{3.8}$$

where the error is $O(\frac{1}{n})$, and where

$$\left.\begin{aligned} \frac{1+a_1^2+a_2^2}{a_2} &= 2 + \frac{(1+\rho_1^2-2T)(1+\rho_2^2-2S)}{\rho_1\rho_2} - \frac{U^2}{\rho_1\rho_2} , \\ \frac{a_1(1+a_2)}{a_2} &= -\frac{(1+\rho_1^2-2T)}{\rho_1} - \frac{(1+\rho_2^2-2S)}{\rho_2} , \end{aligned}\right\} \tag{3.9}$$

and

$$\left.\begin{aligned} \beta_{11} &= \frac{(1-2T)(1-2S)}{\rho_1\rho_2} + 1 - \frac{U^2}{\rho_1\rho_2} , \\ \beta_{12} &= -\frac{(1-2T)}{\rho_1} - \frac{(1+\rho_2^2-2S)}{\rho_2} \\ \beta_{21} &= -\frac{(1-2S)}{\rho_2} - \frac{(1+\rho_1^2-2T)}{\rho_1} . \end{aligned}\right\} \tag{3.10}$$

Following the inversion method of Appendix I, we make the substitution

$$U = \frac{v(r_1+r_2)}{r_1\, r_2} - \frac{T}{r_1} - \frac{S}{r_2} , \qquad (3.11)$$

where $r_1 = \frac{E^*}{C^*}$ and $r_2 = \frac{E^*}{D^*}$. As in Chapter II a_1 and a_2 are implicit functions of T,S and v $(a_1 \equiv a_1(T,S,v),\ a_2 \equiv a_2(T,S,v))$ and hence

$$\frac{\partial^2}{\partial v^2}\left[M^*\left(T,S,\ \frac{v(r_1+r_2) - Tr_2 - Sr_1}{r_1\, r_2} \right)\right]$$

$$\sim \frac{(1-\rho_1^2)^{\frac{1}{2}}(1-\rho_2^2)^{\frac{1}{2}}(\frac{n}{2} - \frac{1}{2})(\frac{n}{2} - \frac{3}{2})a_2^{\frac{n}{2}-\frac{5}{2}}}{(1-\rho_1)(1-\rho_2) \quad (\rho_1\rho_2\ )^{\frac{1}{2}(n-1)}}$$

$$\times \frac{(1-a_1+a_2)^{\frac{1}{2}}(1+a_1+a_2)^{3/2}(1-a_2)}{[a_2(\beta_{11}-a_2)-a_2^2(\beta_{12}-a_1)(\beta_{21}-a_1)]} \left(\frac{\partial a_2}{\partial v}\right)^2 ,$$

where the error is relatively $O(\frac{1}{n})$.

Then using the inversion formula (I.3), the joint density of r_1 and r_2 is

$$h^*(r_1,r_2)$$

$$\sim \frac{(1-\rho_1^2)^{\frac{1}{2}}(1-\rho_2^2)^{\frac{1}{2}}(\frac{n}{2} - \frac{1}{2})(\frac{n}{2} - \frac{3}{2})}{(1-\rho_1)\ (1-\rho_2)(\rho_1\rho_2)^{\frac{1}{2}(n-1)}(r_1+r_2)^2}$$

$$\times \frac{1}{(2\pi i)^2} \iint a_2^{\frac{1}{2}(n-5)} \frac{(1-a_1+a_2)^{\frac{1}{2}}(1+a_1+a_2)^{3/2}(1-a_2)}{[a_2(\beta_{11}-a_2)-a_2^2(\beta_{12}-a_1)(\beta_{21}-a_1)]} \left(\frac{\partial a_2}{\partial v}\right)^2 \Bigg|_{v=0} dS\, dT$$

$$(3.12)$$

where the terms which are relatively $O(\frac{1}{n})$ have been omitted.

To evaluate the integral

$$\frac{1}{(2\pi i)^2} \iint a_2^{\frac{1}{2}(n-5)} \frac{(1-a_1+a_2)^{\frac{1}{2}}(1+a_1+a_2)^{3/2}(1-a_2)}{[a_2(\beta_{11}-a_2)-a_2^2(\beta_{12}-a_1)(\beta_{21}-a_1)]} \left(\frac{\partial a_2}{\partial v}\right)^2 \Bigg|_{v=0} dS\,dT \tag{3.13}$$

we use the bivariate, saddle-point approximation,

$$\frac{1}{(2\pi i)^2} \iint [\psi(z_1,z_2)]^{k^*} \varphi^*(z_1,z_2)dz_2dz_1$$

$$\sim \frac{\varphi^*(\hat{z}_1,\hat{z}_2) \quad [\psi(\hat{z}_1,\hat{z}_2)]^{k^*+1}}{2\pi k^*\{\psi_{11}(\hat{z}_1,\hat{z}_2)\psi_{22}(\hat{z}_1,\hat{z}_2)-[\psi_{12}(\hat{z}_1,\hat{z}_2)]^2\}^{\frac{1}{2}}} \quad , \tag{3.14}$$

where $\psi_{11}(\hat{z}_1,\hat{z}_2)$, $\psi_{22}(\hat{z}_1,\hat{z}_2)$, $\psi_{12}(\hat{z}_1,\hat{z}_2)$, $\hat{z}_1$ and $\hat{z}_2$ are defined in (2.14). In (3.13) T and S correspond to z_1 and z_2 in (3.14), respectively, $a_2|_{v=0}$ to $\psi(z_1,z_2)$, $\frac{1}{2}(n-5)$ to k^* and

$$\left[\frac{(1-a_1+a_2)^{\frac{1}{2}}(1+a_1+a_2)^{3/2}(1-a_2)}{[a_2(\beta_{11}-a_2)-a_2^2(\beta_{12}-a_1)(\beta_{21}-a_1)]} \left(\frac{\partial a_2}{\partial v}\right)^2 \Bigg|_{v=0} \right] \text{to } \varphi^*(z_1,z_2) \ .$$

The paths of integration in the T and S planes are taken as the lines of steepest descent of $a_2|_{v=0} \equiv a_2(T,S)$ such that

$$\frac{\partial a_2(\hat{T},\hat{S})}{\partial \hat{T}} = \frac{\partial a_2(\hat{T},\hat{S})}{\partial \hat{S}} = 0 \ .$$

Now comparing (3.13) and (3.14) with (2.13) and (2.14), it can be seen that

$$\varphi^*(T,S) = (1+a_1+a_2)\ \varphi(T,S)$$

and

$$k^* = \tfrac{1}{2}(n - 5), \quad \text{while} \quad k = \tfrac{1}{2}(n - 4)$$

(note: by replacing n by $(n-1)$ in k we obtain k^*).

Otherwise there are no differences between (3.13) and (2.13), and hence to obtain $\varphi^*(\hat{T},\hat{S})$,

$$\varphi^*(\hat{T},\hat{S}) = (1+\hat{a}_1+\hat{a}_2)\ \varphi(\hat{T},\hat{S}) ,$$

where $\hat{a}_1 \equiv a_1(\hat{T},\hat{S})$ and $\hat{a}_2 \equiv (\hat{T},\hat{S})$, we can simply use the expressions (2.34) and (2.30) derived in Chapter II. Then

$$\left[\frac{(1-\hat{a}_1+\hat{a}_2)^{\frac{1}{2}}(1+\hat{a}_1+\hat{a}_2)^{3/2}(1-\hat{a}_2)}{[\hat{a}_2(\hat{\beta}_{11}-\hat{a}_2)-\hat{a}_2^2(\hat{\beta}_{12}-\hat{a}_1)(\hat{\beta}_{21}-\hat{a}_1)]} \left(\frac{\partial \hat{a}_2}{\partial v}\right)^2 \Bigg|_{v=0} \right] = \varphi^*(\hat{T},\hat{S})$$

$$= \frac{(1+\hat{a}_2)[r_1(1-\rho_2)^2+r_2(1-\rho_1)^2]}{[r_1(1+\rho_2^2)+r_2(1+\rho_1^2)]}$$

$$\times 16\hat{a}_2^2(1-\hat{a}_2)^2(r_1+r_2)^2[r_1(1+\rho_2^2)+r_2(1+\rho_1^2)]^2$$

$$\times \Big[(1+\hat{a}_2)(1-\hat{a}_2\rho_1\rho_2)[r_1(1-\rho_2^2)+r_2(1-\rho_1^2)]\Big]^{-1}$$

$$\times \Big[r_1^2(1-\rho_2^2)^2+r_2^2(1-\rho_1^2)^2+2r_1r_2[(\rho_1-\rho_2)^2+(1-\rho_1\rho_2)^2]\ \Big]^{-3/2}$$

$$= 16\hat{a}_2^2(1-\hat{a}_2)^2(r_1+r_2)^2[r_1(1+\rho_2^2)+r_2(1+\rho_1^2)]$$

$$\times [r_1(1-\rho_2)^2+r_2(1-\rho_1)^2]\Big[(1-\hat{a}_2\rho_1\rho_2)[r_1(1-\rho_2^2)+r_2(1-\rho_1^2)]\Big]^{-1}$$

$$\times \Big[r_1^2(1-\rho_2^2)^2+r_2^2(1-\rho_1^2)^2+2r_1r_2[(\rho_1-\rho_2)^2+(1-\rho_1\rho_2)^2]\Big]^{-3/2} . \tag{3.15}$$

From (2.40), we have

$$\left[\left(\frac{\partial^2 a_2}{\partial T^2}\right)\left(\frac{\partial^2 a_2}{\partial S^2}\right) - \left(\frac{\partial^2 a_2}{\partial T\partial S}\right)^2\right]^{\frac{1}{2}}\Bigg|_{a_2=\hat{a}_2}$$

$$= 2\hat{a}_2^{5/2}[r_1(1+\rho_2^2)+r_2(1+\rho_1^2)]^3[r_1r_2(\rho_1\rho_2)^{3/2}(1-\hat{a}_2)(1+\hat{a}_2)^2]^{-1}$$

$$\times\left\{r_1^2(1-\rho_2^2)^2+r_2^2(1-\rho_1^2)^2+2r_1r_2[(\rho_1-\rho_2)^2+(1-\rho_1\rho_2)^2]\right\}^{-1} \quad .$$

Finally, since $k^* = \frac{1}{2}(n-5)$ and $\psi(z_1,z_2) = a_2(T,S)$, the bivariate saddle-point approximation of (3.13) is

$$\frac{1}{(2\pi i)^2}\iint a_2^{\frac{1}{2}(n-5)} \frac{(1-a_1+a_2)^{\frac{1}{2}}(1+a_1+a_2)^{3/2}(1-a_2)}{[a_2(\beta_{11}-a_2)-a_2^2(\beta_{12}-a_1)(\beta_{21}-a_1)]}\left(\frac{\partial a_2}{\partial v}\right)^2\Bigg|_{v=0} dS\,dT$$

$$\sim \frac{\hat{a}_2^{\frac{1}{2}(n-3)}}{2\pi\frac{1}{2}(n-5)} \cdot \frac{16\hat{a}_2^2(1-\hat{a}_2)^2(r_1+r_2)^2r_1r_2(\rho_1\rho_2)^{3/2}(1-\hat{a}_2)(1+\hat{a}_2)^2}{(1-\hat{a}_2\rho_1\rho_2)2\hat{a}_2^{5/2}}$$

$$\times [r_1(1+\rho_2^2)+r_2(1+\rho_1^2)]^{-2}[r_1(1-\rho_2^2)+r_2(1-\rho_1^2)]^{-1}$$

$$\times [r_1(1-\rho_2)^2+r_2(1-\rho_1)^2]$$

$$\times\left\{r_1^2(1-\rho_2^2)^2+r_2^2(1-\rho_1^2)^2+2r_1r_2[(\rho_1-\rho_2)^2+(1-\rho_1\rho_2)^2]\right\}^{-\frac{1}{2}} \quad .$$

Thus the approximate joint marginal density function of r_1 and r_2 (3.12) becomes

$$h^*(r_1,r_2) \sim \frac{(1-\rho_1^2)^{\frac{1}{2}}(1-\rho_2^2)^{\frac{1}{2}}(\frac{n}{2}-\frac{1}{2})(\frac{n}{2}-\frac{3}{2})}{(1-\rho_1)(1-\rho_2)(\rho_1\rho_2)^{\frac{1}{2}(n-1)}(r_1+r_2)^2}$$

$$\times\frac{8(r_1+r_2)^2r_1r_2(\rho_1\rho_2)^{3/2}(1-\hat{a}_2)^3(1+\hat{a}_2)^2\ \hat{a}_2^{\frac{1}{2}n-2}}{\pi(n-5)(1-\hat{a}_2\rho_1\rho_2)}$$

$$\times [r_1(1+\rho_2^2)+r_2(1+\rho_1^2)]^{-2}[r_1(1-\rho_2^2)+r_2(1-\rho_1^2)]^{-1}$$

$$\times [r_1(1-\rho_2)^2+r_2(1-\rho_1)^2]$$

$$\times\left\{r_1^2(1-\rho_2^2)^2+r_2^2(1-\rho_1^2)^2+2r_1r_2[(\rho_1-\rho_2)^2+(1-\rho_1\rho_2)^2]\right\}^{-\frac{1}{2}} \quad ,$$

and after simplication

$$h^*(r_1,r_2) \sim \frac{2(n-1)(n-3)}{\pi(n-5)} \cdot \frac{(1-\rho_1^2)^{\frac{1}{2}}(1-\rho_2^2)^{\frac{1}{2}}(1-\hat{a}_2)^3(1+\hat{a}_2)^2 r_1 r_2}{(1-\rho_1)(1-\rho_2)(1-\hat{a}_2\rho_1\rho_2)}$$

$$\times \left[\frac{\hat{a}_2}{\rho_1\rho_2}\right]^{\frac{1}{2}n-2} \frac{[r_1(1-\rho_2)^2+r_2(1-\rho_1)^2]}{[r_1(1+\rho_2^2)+r_2(1+\rho_1^2)]}$$

$$\times [r_1(1-\rho_2^2)+r_2(1-\rho_1^2)]^{-1}[r_1(1+\rho_2^2)+r_2(1+\rho_1^2)]^{-1}$$

$$\times\left\{r_1^2(1-\rho_2^2)^2+r_2^2(1-\rho_1^2)^2+2r_1r_2[(\rho_1-\rho_2)^2+(1-\rho_1\rho_2)^2]\right\}^{-\frac{1}{2}} , \qquad (3.16)$$

where the relative error of the approximation is $O(\frac{1}{n})$.

The product-moment correlation r is related to r_1 and r_2 by $r^2 = r_1r_2$. The approximate density function of r can be obtained from (3.16) as a marginal density. First let

$$\zeta = r_1r_2 \qquad \text{and} \qquad \xi = \frac{r_1}{r_2} .$$

Then $r_1^2 = \zeta\xi$, $r_2^2 = \frac{\zeta}{\xi}$ and $\frac{\partial(r_1,r_2)}{\partial(\zeta,\xi)} = \frac{1}{2\xi}$.

Writing the expressions involving r_1 and r_2 in (3.16) in terms of ζ and ξ, we have

$$[r_1(1-\rho_2)^2+r_2(1-\rho_1)^2] = \sqrt{\zeta}\left[\sqrt{\xi}\,(1-\rho_2)^2 + \frac{1}{\sqrt{\xi}}\,(1-\rho_1)^2\right] ,$$

$$[r_1(1+\rho_2^2)+r_2(1+\rho_1^2)] = \sqrt{\zeta}\left[\sqrt{\xi}\,(1+\rho_2^2) + \frac{1}{\sqrt{\xi}}\,(1+\rho_1^2)\right] ,$$

$$[r_1(1-\rho_2^2)+r_2(1-\rho_1^2)] = \sqrt{\zeta}\left[\sqrt{\xi}\,(1-\rho_2^2) + \frac{1}{\sqrt{\xi}}\,(1-\rho_1^2)]\right]$$

and

$$\left\{r_1^2(1-\rho_2^2)^2+r_2^2(1-\rho_1^2)^2+2r_1r_2[(\rho_1-\rho_2)^2+(1-\rho_1\rho_2)^2]\right\}^{\frac{1}{2}}$$

$$=\sqrt{\zeta}\left\{\xi(1-\rho_2^2)^2+\frac{1}{\xi}(1-\rho_1^2)^2+2[(\rho_1-\rho_2)^2+(1-\rho_1\rho_2)^2]\right\}^{\frac{1}{2}} .$$

Thus, substituting the above expressions in (3.16), the approximate joint density of ζ and ξ is

$$g^*(\zeta,\xi) \sim \frac{2(n-1)(n-3)}{\pi(n-5)} \cdot \frac{(1-\rho_1^2)^{\frac{1}{2}}(1-\rho_2^2)^{\frac{1}{2}}(1-\hat{a}_2)^3(1+\hat{a}_2)^2\,\zeta}{(1-\rho_1)(1-\rho_2)(1-\hat{a}_2\rho_1\rho_2)\,2\xi}$$

$$\times\left[\frac{\hat{a}_2}{\rho_1\rho_2}\right]^{\frac{1}{2}n-2} \cdot \frac{[\sqrt{\xi}\,(1-\rho_2)^2+\frac{1}{\sqrt{\xi}}\,(1-\rho_1)^2]}{[\sqrt{\xi}\,(1+\rho_2^2)+\frac{1}{\sqrt{\xi}}\,(1+\rho_1^2)]}$$

$$\times\left[\zeta\,[\sqrt{\xi}\,(1-\rho_2^2)+\frac{1}{\sqrt{\xi}}\,(1-\rho_1^2)][\sqrt{\xi}\,(1+\rho_2^2)+\frac{1}{\sqrt{\xi}}\,(1+\rho_1^2)]\right]^{-1}$$

$$\times\ \zeta^{-\frac{1}{2}}\left[\xi(1-\rho_2^2)^2+\frac{1}{\xi}(1-\rho_1^2)^2+2[(\rho_1-\rho_2)^2+(1-\rho_1\rho_2)^2]\right]^{-\frac{1}{2}} , \qquad (3.17)$$

where $\hat{a}_2$ is now a function of ζ and ξ .

Then the approximate marginal density function of ζ is

$$f^*(\zeta)=\int_\xi g^*(\zeta,\xi)d\xi \sim \int [\chi(\zeta,\xi)]^{m^*}\ \theta^*(\zeta,\xi)d\xi , \qquad (3.18)$$

where from (3.17)

$$\chi(\zeta,\xi) = \frac{\hat{a}_2}{\rho_1\rho_2} , \qquad m^* = \tfrac{1}{2}n-2 \qquad (3.19)$$

and

$$\theta^*(\zeta,\xi) = \frac{(n-1)(n-3)(1-\rho_1^2)^{\frac{1}{2}}(1-\rho_2^2)^{\frac{1}{2}}}{\pi(n-5)(1-\rho_1)(1-\rho_2)} \cdot \frac{(1-\hat{a}_2)^3(1+\hat{a}_2)^2}{(1-\hat{a}_2\rho_1\rho_2)\ \xi\ \zeta^{\frac{1}{2}}}$$

$$\times \frac{[\sqrt{\xi}\ (1-\rho_2)^2 + \frac{1}{\sqrt{\xi}}\ (1-\rho_1)^2]}{[\sqrt{\xi}\ (1+\rho_2^2) + \frac{1}{\sqrt{\xi}}\ (1+\rho_1^2)]}$$

$$\times [\sqrt{\xi}\ (1-\rho_2^2) + \frac{1}{\sqrt{\xi}}\ (1-\rho_1^2)]^{-1}\ [\sqrt{\xi}\ (1+\rho_2^2) + \frac{1}{\sqrt{\xi}}\ (1+\rho_1^2)]^{-1}$$

$$\times\Big[\xi\ (1-\rho_2^2)^2 + \frac{1}{\xi}\ (1-\rho_1^2)^2+2[(\rho_1-\rho_2)^2 + (1-\rho_1\rho_2)^2]\Big]^{-\frac{1}{2}} . \qquad (3.20)$$

Following Daniels [4], the saddle-point approximation of (3.18) is

$$f^*(\zeta) \sim \left[\frac{-2\pi\ \chi(\zeta,\hat{\xi})}{m^*\chi''(\zeta,\hat{\xi})}\right]^{\frac{1}{2}} \quad \theta^*(\zeta,\hat{\xi}) \left[\chi(\zeta,\hat{\xi})\ \right]^{m^*} , \qquad (3.21)$$

where, as in (2.45), $\hat{\xi}$ is the solution of

$$\chi'(\zeta,\xi) = \frac{\partial\chi(\zeta,\xi)}{\partial\xi} = 0$$

and

$$\chi''(\zeta,\hat{\xi}) = \frac{\partial^2\chi(\zeta,\xi)}{\partial\xi^2}\ \Bigg|_{\xi=\hat{\xi}} .$$

Comparing (3.20) with (2.44), we observe that

$$\theta^*(\zeta,\xi) = \frac{(n-1)(n-3)(n-4)}{(n-5)\ n(n-2)} \cdot \frac{(1+\hat{a}_2)}{(1-\rho_1)(1-\rho_2)}$$

$$\times \frac{[\sqrt{\xi}\ (1-\rho_2)^2 + \frac{1}{\sqrt{\xi}}\ (1-\rho_1)^2]}{[\sqrt{\xi}\ (1+\rho_2^2) + \frac{1}{\sqrt{\xi}}\ (1+\rho_1^2)]} \ .\ \theta(\zeta,\xi) , \qquad (3.22)$$

$$m^* = \tfrac{1}{2}n-2 , \quad \text{whereas} \quad m = \tfrac{1}{2}(n-3)$$

and

$$\chi(\zeta,\xi) = \frac{\hat{a}_2}{\rho_1\rho_2}$$ is the same as in (2.44).

Hence from (2.49)

$$\hat{\xi} = \frac{1-\rho_1^2}{1-\rho_2^2} , \tag{3.23}$$

from (2.51)

$$\chi(\zeta,\hat{\xi}) = 2(1-\zeta)\left\{1+\rho_1^2\rho_2^2-2\rho_1\rho_2\ \zeta+(1-\rho_1\rho_2)[(1+\rho_1\rho_2)^2-4\rho_1\rho_2\zeta]^{\frac{1}{2}}\right\}^{-1} \tag{3.24}$$

and from (2.52)

$$\frac{-\chi(\zeta,\hat{\xi})}{\chi''(\zeta,\hat{\xi})} = \frac{2(1-\rho_1^2)(1-\rho_1\rho_2)[(1+\rho_1\rho_2)^2-4\rho_1\rho_2\zeta]^{\frac{1}{2}}}{(1-\rho_2^2)^3} . \tag{3.25}$$

Substituting $\hat{\xi}$ for ξ in (3.22), $\theta^*(\zeta,\hat{\xi})$ becomes

$$\theta^*(\zeta,\hat{\xi}) = \frac{(n-1)(n-3)(n-4)}{(n-5)\ n(n-2)} \cdot \frac{[1+\rho_1\rho_2\chi(\zeta,\hat{\xi})]}{(1-\rho_1)(1-\rho_2)}$$

$$\times \frac{[\sqrt{\hat{\xi}}\ (1-\rho_2)^2 + \frac{1}{\sqrt{\hat{\xi}}}\ (1-\rho_1)^2]}{[\sqrt{\hat{\xi}}\ (1+\rho_2^2) + \frac{1}{\sqrt{\hat{\xi}}}\ (1+\rho_1^2)]} \cdot \theta(\zeta,\hat{\xi}) . \tag{3.26}$$

Using (3.23), we may write

$$[\sqrt{\hat{\xi}}\ (1-\rho_2)^2 + \frac{1}{\sqrt{\hat{\xi}}}\ (1-\rho_1)^2] = \frac{(1-\rho_1^2)^{\frac{1}{2}}}{(1-\rho_2^2)^{\frac{1}{2}}}\ (1-\rho_2)^2 + \frac{(1-\rho_2^2)^{\frac{1}{2}}}{(1-\rho_1^2)^{\frac{1}{2}}}\ (1-\rho_1)^2$$

$$= \frac{[(1-\rho_1^2)(1-\rho_2)^2 + (1-\rho_2^2)(1-\rho_1)^2]}{(1-\rho_1^2)^{\frac{1}{2}}\ (1-\rho_2^2)^{\frac{1}{2}}}$$

$$= \frac{(1-\rho_1)(1-\rho_2)[(1+\rho_1)(1-\rho_2) + (1+\rho_2)(1-\rho_1)]}{(1-\rho_1^2)^{\frac{1}{2}}\ (1-\rho_2^2)^{\frac{1}{2}}}$$

$$= \frac{2(1-\rho_1)(1-\rho_2)(1-\rho_1\rho_2)}{(1-\rho_1^2)^{\frac{1}{2}}\ (1-\rho_2^2)^{\frac{1}{2}}} .$$

From (2.55) we have

$$[\sqrt{\hat{\xi}}\ (1+\rho_2^2) + \frac{1}{\sqrt{\hat{\xi}}}\ (1+\rho_1^2)] = \frac{2(1-\rho_1\rho_2)(1+\rho_1\rho_2)}{(1-\rho_1^2)^{\frac{1}{2}}\ (1-\rho_2^2)^{\frac{1}{2}}}$$

Thus

$$\frac{[\sqrt{\hat{\xi}}\ (1-\rho_2)^2 + \frac{1}{\sqrt{\hat{\xi}}}\ (1-\rho_1)^2]}{[\sqrt{\hat{\xi}}\ (1+\rho_2^2) + \frac{1}{\sqrt{\hat{\xi}}}\ (1+\rho_1^2)]} = \frac{(1-\rho_1)(1-\rho_2)}{(1+\rho_1\rho_2)} \quad . \tag{3.27}$$

From (2.54)

$$[1+\rho_1\rho_2\chi(\zeta,\hat{\xi})]$$

$$= \frac{[(1+\rho_1\rho_2)^2-4\rho_1\rho_2\zeta]^{\frac{1}{2}}\left\{1-\rho_1\rho_2+[(1+\rho_1\rho_2)^2-4\rho_1\rho_2\zeta]^{\frac{1}{2}}\right\}}{\left\{1+\rho_1^2\rho_2^2-2\rho_1\rho_2\zeta + (1-\rho_1\rho_2)[(1+\rho_1\rho_2)^2 - 4\rho_1\rho_2\zeta]^{\frac{1}{2}}\right\}} \tag{3.28}$$

and from (2.56)

$$\theta(\zeta,\hat{\xi}) = \frac{n(n-2)(1-\rho_2^2)^{3/2}}{8\pi(n-4)(1-\rho_1^2)^{\frac{1}{2}}(1+\rho_1\rho_2)} \cdot \frac{[(1+\rho_1\rho_2)^2 - 4\rho_1\rho_2\zeta]^{\frac{1}{2}}}{\zeta^{\frac{1}{2}}}$$

$$\times\left[1-\rho_1\rho_2+[(1+\rho_1\rho_2)^2-4\rho_1\rho_2\zeta]^{\frac{1}{2}}\right]^4$$

$$\times\left[1+\rho_1^2\rho_2^2-2\rho_1\rho_2\zeta+(1-\rho_1\rho_2)[(1+\rho_1\rho_2)^2-4\rho_1\rho_2\zeta]^{\frac{1}{2}}\right]^{-3}$$

$$\times\left[1+\rho_1\rho_2(1-2\zeta)+[(1+\rho_1\rho_2)^2-4\rho_1\rho_2\zeta]^{\frac{1}{2}}\right]^{-1} \quad . \tag{3.29}$$

Then using (3.27), (3.28) and (3.29) in (3.26), we have after simplication

$$\theta^*(\zeta,\hat{\xi}) = \frac{(n-1)(n-3)(1-\rho_2^2)^{3/2}}{8\pi(n-5)(1-\rho_1^2)^{\frac{1}{2}}(1+\rho_1\rho_2)^2} \cdot \frac{[(1+\rho_1\rho_2)^2-4\rho_1\rho_2\zeta]}{\zeta^{\frac{1}{2}}}$$

$$\times\left[1-\rho_1\rho_2+[(1+\rho_1\rho_2)^2-4\rho_1\rho_2\zeta]^{\frac{1}{2}}\right]^5$$

$$\times\left[1+\rho_1^2\rho_2^2-2\rho_1\rho_2\zeta+(1-\rho_1\rho_2)[(1+\rho_1\rho_2)^2-4\rho_1\rho_2\zeta]^{\frac{1}{2}}\right]^{-4}$$

$$\times\left[1+\rho_1\rho_2(1-2\zeta)+[(1+\rho_1\rho_2)^2-4\rho_1\rho_2\zeta]^{\frac{1}{2}}\right]^{-1} . \tag{3.30}$$

Now substituting (3.24), (3.25) and (3.30) in (3.21), we obtain

$$f^*(\zeta) \sim \left[\frac{2\pi}{(\frac{1}{2}n-2)} \frac{2(1-\rho_1^2)(1-\rho_1\rho_2)[(1+\rho_1\rho_2)^2-4\rho_1\rho_2\zeta]^{\frac{1}{2}}}{(1-\rho_2^2)^3}\right]^{\frac{1}{2}}$$

$$\times\frac{(n-1)(n-3)(1-\rho_2^2)^{3/2} \quad [(1+\rho_1\rho_2)^2-4\rho_1\rho_2\zeta]}{8\pi(n-5)(1-\rho_1^2)^{\frac{1}{2}} \quad (1+\rho_1\rho_2)^2 \quad \zeta^{\frac{1}{2}}}$$

$$\times\left[1-\rho_1\rho_2+[(1+\rho_1\rho_2)^2-4\rho_1\rho_2\zeta]^{\frac{1}{2}}\right]^5$$

$$\times\left[1+\rho_1^2\rho_2^2-2\rho_1\rho_2\zeta+(1-\rho_1\rho_2)[(1+\rho_1\rho_2)^2-4\rho_1\rho_2\zeta]^{\frac{1}{2}}\right]^{-4}$$

$$\times\left[1+\rho_1\rho_2(1-2\zeta)+[(1+\rho_1\rho_2)^2-4\rho_1\rho_2\zeta]^{\frac{1}{2}}\right]^{-1}$$

$$\times 2^{\frac{1}{2}n-2}\ (1-\zeta)^{\frac{1}{2}n-2}$$

$$\times\left[1+\rho_1^2\rho_2^2-2\rho_1\rho_2\zeta+(1-\rho_1\rho_2)[(1+\rho_1\rho_2)^2-4\rho_1\rho_2\zeta]^{\frac{1}{2}}\right]^{-\frac{1}{2}n+2}$$

which, when simplified, yields

$$f^*(\zeta) \sim \frac{(n-1)(n-3)\ 2^{\frac{1}{2}n-3}\ (1-\rho_1\rho_2)^{\frac{1}{2}}\ (1-\zeta)^{\frac{1}{2}n-2}\ [(1+\rho_1\rho_2)^2-4\rho_1\rho_2\zeta]^{5/4}}{\sqrt{2\pi}\ (n-5)\ \sqrt{n-4}\ \ (1+\rho_1\rho_2)^2\ \ \ \zeta^{\frac{1}{2}}}$$

$$\times\left[1-\rho_1\rho_2+[(1+\rho_1\rho_2)^2-4\rho_1\rho_2\zeta]^{\frac{1}{2}}\right]^5$$

$$\times\left[1+\rho_1^2\rho_2^2-2\rho_1\rho_2\zeta+(1-\rho_1\rho_2)[(1+\rho_1\rho_2)^2-4\rho_1\rho_2\zeta]^{\frac{1}{2}}\right]^{-(\frac{1}{2}n+2)}$$

$$\times\left[1+\rho_1\rho_2(1-2\zeta)+[(1+\rho_1\rho_2)^2-4\rho_1\rho_2\zeta]^{\frac{1}{2}}\right]^{-1} \quad . \tag{3.31}$$

Finally, letting $\zeta = r^2$, $d\zeta = 2\sqrt{\zeta}\ dr$, the approximate density function of the product-moment correlation r is

$$p^*(r) \sim \frac{2^{\frac{1}{2}n-2}\ (n-1)(n-3)(1-\rho_1\rho_2)^{\frac{1}{2}}\ (1-r^2)^{\frac{1}{2}n-2}}{\sqrt{2\pi}\ (n-5)\ \sqrt{n-4}\ \ (1+\rho_1\rho_2)^2}$$

$$\times[(1+\rho_1\rho_2)^2-4\rho_1\rho_2r^2]^{5/4}\left[1-\rho_1\rho_2+[(1+\rho_1\rho_2)^2-4\rho_1\rho_2r^2]^{\frac{1}{2}}\right]^5$$

$$\times\left[1+\rho_1^2\rho_2^2-2\rho_1\rho_2r^2+(1-\rho_1\rho_2)[(1+\rho_1\rho_2)^2-4\rho_1\rho_2r^2]^{\frac{1}{2}}\right]^{-(\frac{1}{2}n+2)}$$

$$\times\left[1+\rho_1\rho_2(1-2r^2)+[(1+\rho_1\rho_2)^2-4\rho_1\rho_2r^2]^{\frac{1}{2}}\right]^{-1} \quad . \tag{3.32}$$

Using the relations (2.59) and (2.60), we may write (3.32) as

$$p^*(r) \sim \frac{2^{\frac{1}{2}n-2}\ (n-1)(n-3)(1-\rho_1\rho_2)^{\frac{1}{2}}(1-r^2)^{\frac{1}{2}n-2}}{\sqrt{2\pi}\ (n-5)\ \sqrt{n-4}\ (1+\rho_1\rho_2)^2}$$

$$\times[(1+\rho_1\rho_2)^2-4\rho_1\rho_2r^2]^{5/4}\left[1-\rho_1\rho_2+[(1+\rho_1\rho_2)^2-4\rho_1\rho_2r^2]^{\frac{1}{2}}\right]^5$$

$$\times 2^{\frac{1}{2}n+2}\left[1-\rho_1\rho_2+[(1+\rho_1\rho_2)^2-4\rho_1\rho_2r^2]^{\frac{1}{2}}\right]^{-(n+4)}$$

$$\times 2\left[[(1+\rho_1\rho_2)^2-4\rho_1\rho_2r^2]^{\frac{1}{2}}+(1-\rho_1\rho_2)\right]^{-1}$$

$$\times\left[[(1+\rho_1\rho_2)^2-4\rho_1\rho_2r^2]^{\frac{1}{2}}+(1+\rho_1\rho_2)\right]^{-1}$$

or

$$p^*(r) \sim \frac{2^{n+1}(n-1)(n-3)(1-\rho_1\rho_2)^{\frac{1}{2}}\ (1-r^2)^{\frac{1}{2}n-2}}{\sqrt{2\pi}\ (n-5)\ \sqrt{n-4}\ \ (1+\rho_1\rho_2)^2}$$

$$\times[(1+\rho_1\rho_2)^2-4\rho_1\rho_2r^2]^{5/4}\left[[(1+\rho_1\rho_2)^2-4\rho_1\rho_2r^2]^{\frac{1}{2}}+(1-\rho_1\rho_2)\right]^{-n}$$

$$\times\left[[(1+\rho_1\rho_2)^2-4\rho_1\rho_2r^2]^{\frac{1}{2}} + (1+\rho_1\rho_2)\right]^{-1} .$$

Since the approximate distribution depends only on the product of the serial correlations, let $\alpha = \rho_1\rho_2$, then

$$p^*(r) \sim \frac{2^{n+1}(n-1)(n-3)(1-\alpha)^{\frac{1}{2}}(1-r^2)^{\frac{1}{2}n-2}}{\sqrt{2\pi}(n-5)\sqrt{n-4}\ (1+\alpha)^2}$$

$$\times\frac{[(1+\alpha)^2-4\alpha r^2]^{5/4}}{\left[[(1+\alpha)^2-4\alpha r^2]^{\frac{1}{2}}+(1-\alpha)\right]^n\left[[(1+\alpha)^2-4\alpha r^2]^{\frac{1}{2}}+(1+\alpha)\right]} , \tag{3.33}$$

where the error of the approximation is relatively $O(\frac{1}{n})$. We note that the right member of (3.33) is an even function of r and hence to the order considered, the mean and all of the odd moments are zero.

From the approximate density function $p^*(r)$ (3.33), we may find the approximation of the variance of r. From (2.62) we can write

$$(1-r^2) = \frac{1}{4\alpha}\left[[(1+\alpha)^2-4\alpha r^2]^{\frac{1}{2}}-(1-\alpha)\right]\left[[(1+\alpha)^2-4\alpha r^2]^{\frac{1}{2}}+(1-\alpha)\right]$$

so that (3.33) becomes

$$p^*(r) \sim \frac{2^5(n-1)(n-3)(1-\alpha)^{\frac{1}{2}}\ \ [(1+\alpha)^2-4\alpha r^2]^{5/4}}{\sqrt{2\pi}\ (n-5)\ \sqrt{n-4}\ (1+\alpha)^2\ \alpha^{\frac{1}{2}n-2}}$$

$$\times\frac{\left[[(1+\alpha)^2\ -\ 4\alpha r^2]^{\frac{1}{2}}\ -(1-\alpha)\right]^{\frac{1}{2}n-2}}{\left[[(1+\alpha)^2-4\alpha r^2]^{\frac{1}{2}}+(1-\alpha)\right]^{\frac{1}{2}n+2}\left[[(1+\alpha)^2-4\alpha r^2]^{\frac{1}{2}}+(1+\alpha)\right]} ,$$

or,

$$p^*(r) \sim \frac{2^5(n-1)(n-3)(1-\alpha)^{\frac{1}{2}}}{\sqrt{2\pi}\ (n-5)\sqrt{n-4}\ (1+\alpha)^2\ \alpha^{\frac{1}{2}n-2}}$$

$$\times\left[\frac{[(1+\alpha)^2-4\alpha r^2]^{\frac{1}{2}}-(1-\alpha)}{[(1+\alpha)^2-4\alpha r^2]^{\frac{1}{2}}+(1-\alpha)}\right]^{\frac{1}{2}n-2}$$

$$\times\frac{[(1+\alpha)^2-4\alpha r^2]^{5/4}}{\left[[(1+\alpha)^2-4\alpha r^2]^{\frac{1}{2}}+(1-\alpha)\right]^4\left[[(1+\alpha)^2-4\alpha r^2]^{\frac{1}{2}}+(1+\alpha)\right]} \quad .$$

Then the variance of r is approximately

$$\operatorname{var}(r) \sim \frac{2^5(n-1)(n-3)(1-\alpha)^{\frac{1}{2}}}{\sqrt{2\pi}\ (n-5)\sqrt{n-4}\ (1+\alpha)^2\ \alpha^{\frac{1}{2}n-2}}$$

$$\times\int_{-1}^{1}\left[\frac{[(1+\alpha)^2-4\alpha r^2]^{\frac{1}{2}}-(1-\alpha)}{[(1+\alpha)^2-4\alpha r^2]^{\frac{1}{2}}+(1-\alpha)}\right]^{\frac{1}{2}n-2}\ [(1+\alpha)^2-4\alpha r^2]^{5/4}$$

$$\times\left[[(1+\alpha)^2-4\alpha r^2]^{\frac{1}{2}}+(1-\alpha)\right]^{-4}$$

$$\times\left[[(1+\alpha)^2-4\alpha r^2]^{\frac{1}{2}}+(1+\alpha)\right]^{-1} r^2 dr \quad . \tag{3.34}$$

Denoting the integral of (3.34) by I and using (2.64), we may write,

$$I \sim \int_{-1}^{1}\exp\left[(\tfrac{1}{2}n-2)[\ln\alpha - \frac{1-\alpha}{1+\alpha}r^2]\right]\ [(1+\alpha)^2-4\alpha r^2]^{5/4}$$

$$\times\left[[(1+\alpha)^2-4\alpha r^2]^{\frac{1}{2}} + (1-\alpha)\right]^{-4}$$

$$\times\left[[(1+\alpha)^2-4\alpha r^2]^{\frac{1}{2}} + (1+\alpha)\right]^{-1} r^2 dr \quad . \tag{3.35}$$

Again, following Daniels ([3], equation (3.1)), (3.35) becomes

$$I \sim \left[\frac{2\pi}{(n-4)\ \frac{1-\alpha}{1+\alpha}}\right]^{\frac{1}{2}} \cdot \left[\frac{1}{2(n-4)\ \frac{1-\alpha}{1+\alpha}}\right] \cdot \frac{(1+\alpha)^{3/2}}{2^4}\ \alpha^{\frac{1}{2}n-2} \quad .$$

Therefore, the variance of r is approximately

$$\operatorname{var}(r) \sim \frac{2^5(n-1)(n-3)(1-\alpha)^{\frac{1}{2}}}{\sqrt{2\pi}\,(n-5)\sqrt{n-4}\,(1+\alpha)^2\,\alpha^{\frac{1}{2}n-2}}$$

$$\times \frac{\sqrt{2\pi}\,(1+\alpha)^3\,\alpha^{\frac{1}{2}n-2}}{\sqrt{n-4}\,(n-4)(1-\alpha)^{3/2}\,2^5} ,$$

that is

$$\operatorname{var}(r) \sim \frac{(n-1)(n-3)}{(n-5)(n-4)^2}\,\frac{(1+\alpha)}{(1-\alpha)}$$

and hence

$$\operatorname{var}(r) \sim \frac{1}{n}\left(\frac{1+\alpha}{1-\alpha}\right) , \qquad (3.36)$$

where the error is relatively $O(\frac{1}{n})$. Thus, using the density function $p^*(r)$, which is derived in this chapter, we have verified Bartlett's [2] approximation of the variance of r.

To renormalize the approximate density function $p^*(r)$, we write (3.33) as

$$p^*(r) \sim K\,\frac{(1-r^2)^{\frac{1}{2}n-2}\,[(1+\alpha)^2-4\alpha r^2]^{5/4}}{\left[[(1+\alpha)^2-4\alpha r^2]^{\frac{1}{2}}+(1-\alpha)\right]^n\left[[(1+\alpha)^2-4\alpha r^2]^{\frac{1}{2}}+(1+\alpha)\right]} , \qquad (3.37)$$

where $$K = \frac{2^{n+1}(n-1)(n-3)(1-\alpha)^{\frac{1}{2}}}{\sqrt{2\pi}\,(n-5)\sqrt{n-4}\,(1+\alpha)^2} .$$

In order to be able to use Appendix IV we want to express $p^*(r)$ in the form

$$p^*(r) \sim K\,\frac{(1-r^2)^s\,\left[[(1+\alpha)^2-4\alpha r^2]^{\frac{1}{2}}+(1+\alpha)\,\right]^{\frac{1}{2}}}{\left[[(1+\alpha)^2-4\alpha r^2]^{\frac{1}{2}}+(1-\alpha)\right]^t\,[(1+\alpha)^2-4\alpha r^2]^{\frac{1}{2}}} .$$

Thus, let us adjust the exponents of (3.37) so that

$$p^*(r) \sim K \frac{(1-r^2)^{\frac{1}{2}n-2}[(1+\alpha)^2-4\alpha r^2]^{7/4}}{\left[[(1+\alpha)^2-4\alpha r^2]^{\frac{1}{2}}+(1+\alpha)\right]^{3/2}}$$

$$\times\frac{\left[[(1+\alpha)^2-4\alpha r^2]^{\frac{1}{2}}+(1-\alpha)\right]^{\delta}}{\left[[(1+\alpha)^2-4\alpha r^2]^{\frac{1}{2}}+(1-\alpha)\right]^{n+\delta}}$$

$$\times\frac{\left[[(1+\alpha)^2-4\alpha r^2]^{\frac{1}{2}}+(1+\alpha)\right]^{\frac{1}{2}}}{[(1+\alpha)^2-4\alpha r^2]^{\frac{1}{2}}} \quad . \tag{3.38}$$

As in Chapter II, we expand the following functions of r^2 as Maclaurin series, in order to obtain the most convenient δ :

for $\quad g_1(r^2) = [(1+\alpha)^2-4\alpha r^2]^{7/4}$, using (2.69), we have

$$g_1(r^2) = (1+\alpha)^{7/2}\left[1-r^2\right]^{\frac{7\alpha}{(1+\alpha)^2}} + O(r^4) \quad , \tag{3.39}$$

for $\quad g_2(r^2) = \left[[(1+\alpha)^2-4\alpha r^2]^{\frac{1}{2}} + (1+\alpha)\right]^{-3/2}$, using (2.70), we have

$$g_2(r^2) = [2(1+\alpha)]^{-3/2}\left[1-r^2\right]^{\frac{-3\alpha}{2(1+\alpha)^2}} + O(r^4) \tag{3.40}$$

and for $\quad g_3(r^2) = \left[[(1+\alpha)^2-4\alpha r^2]^{\frac{1}{2}} + (1-\alpha)\right]^{\delta}$, using (2.71), we have

$$g_3(r^2) = 2^{\delta}\left[1-r^2\right]^{\frac{\alpha\delta}{(1+\alpha)}} + O(r^4) \ . \tag{3.41}$$

Since the expected value of r^2 is $O(n^{-1})$, the error involved in substituting (3.39), (3.40) and (3.41) in (3.38) and omitting the $O(r^4)$ terms is relatively $O(n^{-2})$. Thus we have

$$p^*(r) \sim \frac{K(1+\alpha)^{7/2}\, 2^{\delta}}{2^{3/2}(1+\alpha)^{3/2}}$$

$$\times \frac{[1-r^2]^{\frac{1}{2}n-2+\frac{7\alpha}{(1+\alpha)^2}-\frac{3\alpha}{2(1+\alpha)^2}+\frac{\alpha\delta}{(1+\alpha)}}}{\left[[(1+\alpha)^2-4\alpha r^2]^{\frac{1}{2}}+(1-\alpha)\right]^{n+\delta}}$$

$$\times \frac{\left[[(1+\alpha)^2-4\alpha r^2]^{\frac{1}{2}}+(1+\alpha)\right]^{\frac{1}{2}}}{[(1+\alpha)^2-4\alpha r^2]^{\frac{1}{2}}} \quad ,$$

which may be written as

$$p^*(r) \sim K' \frac{\left[1-r^2\right]^{\frac{1}{2}\left[n-4+\frac{11\alpha}{(1+\alpha)^2}+\frac{2\alpha\delta}{(1+\alpha)}\right]}}{\left[[(1+\alpha)^2-4\alpha r^2]^{\frac{1}{2}}+(1-\alpha)\right]^{n+\delta}}$$

$$\times \frac{\left[[(1+\alpha)^2-4\alpha r^2]^{\frac{1}{2}}+(1+\alpha)\right]^{\frac{1}{2}}}{[(1+\alpha)^2-4\alpha r^2]^{\frac{1}{2}}} \quad , \qquad (3.42)$$

where $K' = K(1+\alpha)^2\, 2^{\delta-3/2}$. Now we choose δ such that

$$n+\delta = 2[\text{exponent of } (1-r^2)] + \frac{3}{2} \quad ,$$

that is

$$n+\delta = n-4+\frac{11\alpha}{(1+\alpha)^2}+\frac{2\alpha\delta}{(1+\alpha)}+\frac{3}{2}$$

or

$$\delta\left[1-\frac{2\alpha}{1+\alpha}\right] = \frac{11\alpha}{(1+\alpha)^2}-\frac{5}{2} \quad .$$

Hence

$$\delta\left[\frac{1-\alpha}{1+\alpha}\right] = -\frac{[5\alpha^2-12\alpha+5]}{2(1+\alpha)^2}$$

and thus

$$\delta = -\frac{[5\alpha^2-12\alpha+5]}{2(1+\alpha)(1-\alpha)} \quad . \qquad (3.43)$$

Also let

$$[\text{exponent of } (1-r^2)] = \tfrac{1}{2} M-2 \quad ,$$

that is

$$M = n + \frac{11\alpha}{(1+\alpha)^2} + \frac{2\alpha\delta}{(1+\alpha)} \quad .$$

Using (3.43),

$$M = n + \frac{11\alpha}{(1+\alpha)^2} - \frac{\alpha[5\alpha^2 - 12\alpha + 5]}{(1+\alpha)^2 \ (1-\alpha)}$$

$$= n - \frac{[5\alpha^3 - \alpha^2 - 6\alpha]}{(1+\alpha)^2 \ (1-\alpha)}$$

$$= n + \frac{\alpha(6 - 5\alpha)}{1 - \alpha^2} \quad ,$$

and thus

$$n + \delta = 2[\tfrac{1}{2}M - 2] + \tfrac{3}{2} = M - \tfrac{5}{2} \quad .$$

Then (3.42) may be written as

$$p^*(r) \sim K' \frac{(1-r^2)^{\frac{1}{2}M-2}}{\left[[(1+\alpha)^2-4\alpha r^2]^{\frac{1}{2}}+(1-\alpha)\right]^{M-5/2}} \times \frac{\left[[(1+\alpha)^2-4\alpha r^2]^{\frac{1}{2}}+(1+\alpha)\right]^{\frac{1}{2}}}{[(1+\alpha)^2-4\alpha r^2]^{\frac{1}{2}}} \quad . \tag{3.44}$$

For the renormalization of $p^*(r)$ we must evaluate the integral

$$\int_{-1}^{1} p^*(r)\, dr = 2\int_{0}^{1} p^*(r)\, dr \quad .$$

Using (3.44),

$$\int_0^1 p^*(r)dr \sim K^* \int_0^1 \frac{(1-r^2)^{\frac{1}{2}M-2}}{\left[[(1+\alpha)^2-4\alpha r^2]^{\frac{1}{2}}+(1-\alpha)\right]^{M-5/2}}$$

$$\times \frac{\left[[(1+\alpha)^2-4\alpha r^2]^{\frac{1}{2}}+(1+\alpha)\right]^{\frac{1}{2}}}{[(1+\alpha)^2-4\alpha r^2]^{\frac{1}{2}}} \, dr \quad ,$$

where K^* is chosen to make the right member equal to $\frac{1}{2}$.

Now we may use Appendix IV, since $\int_0^1 p^*(r)\, dr$ is in the desired form, and hence

$$\int_0^1 p^*(r)dr \sim K^* \, J[\tfrac{1}{2}M-2,\ M-\tfrac{5}{2},\ \alpha] \quad .$$

By construction we set

$$M - \frac{5}{2} = 2(\tfrac{1}{2}M - 2) + \frac{3}{2} \quad ,$$

and thus we may use the relation (IV.13) to give

$$2\int_0^1 p^*(r)dr \sim K^* 2^{-(M-2)+1} \, B(\tfrac{1}{2}M-1,\tfrac{1}{2}) \, (1-\alpha)^{-\frac{1}{2}} \quad . \qquad (3.45)$$

Thus $K^* = \dfrac{2^{M-3}(1-\alpha)^{\frac{1}{2}}}{B(\frac{1}{2}M-1,\frac{1}{2})}$. Finally, the approximate density function of the product-moment correlation r in renormalized form is

$$p^*(r) = \frac{2^{M-3}\,(1-\alpha)^{\frac{1}{2}}}{B(\frac{1}{2}\,M-1,\frac{1}{2})} \cdot \frac{(1-r^2)^{\frac{1}{2}M-2}}{\left[[(1+\alpha)^2-4\alpha r^2]^{\frac{1}{2}} + (1-\alpha)\right]^{M-5/2}}$$

$$\times \frac{\left[[(1+\alpha)^2-4\alpha r^2]^{\frac{1}{2}} + (1+\alpha)\right]^{\frac{1}{2}}}{[(1+\alpha)^2 - 4\alpha r^2]^{\frac{1}{2}}} \quad \left\{ 1 + O(n^{-1}) \right\}, \qquad (3.46)$$

where $M = n + \dfrac{\alpha(6 - 5\alpha)}{(1-\alpha^2)}$, $\alpha = \rho_1\rho_2$. Hence, it can be seen that, to the order considered the approximate density function, $p^*(r)$, of the

sample product-moment correlation r depends only on the product of the serial correlations, ρ_1 and ρ_2, and that for $\alpha = \rho_1\rho_2 = 0$, $p^*(r)$, (3.46), reduces to the familiar null distribution of the product-moment correlation with fitted means, (see Keeping (11.12.3) in [7]

$$p^*(r) = \frac{(1-r^2)^{\frac{1}{2}(n-4)}}{B[\frac{1}{2}(n-2),\frac{1}{2}]} \quad) \quad .$$

We also note that by replacing N by $(M-1)$ in $p(r)$, (2.76), the approximate density function of r for the known-means case, we obtain the approximate density function of r for the fitted-means case, namely, $p^*(r)$, (3.46) .

For a graphical representation of $p^*(r)$ (3.46), see Figure 1 below. For the calculations Peters' [14] and Davis'[5] tables were used.

Table of Values for Figure 1

$\pm r$	$p^*(r)$		
	α=-0.5	α=0	α=0.5
0	3.234	2.092	1.260
.1	2.293	1.836	1.212
.2	.867	1.231	1.072
.3	.198	.614	.864
.4	.030	.217	.618
.5	.003	.050	.374
.6	.000	.006	.176
.7	.000	.000	.053
.8	.000	.000	.007
.9	.000	.000	.000
1.0	.000	.000	.000

TABLE I

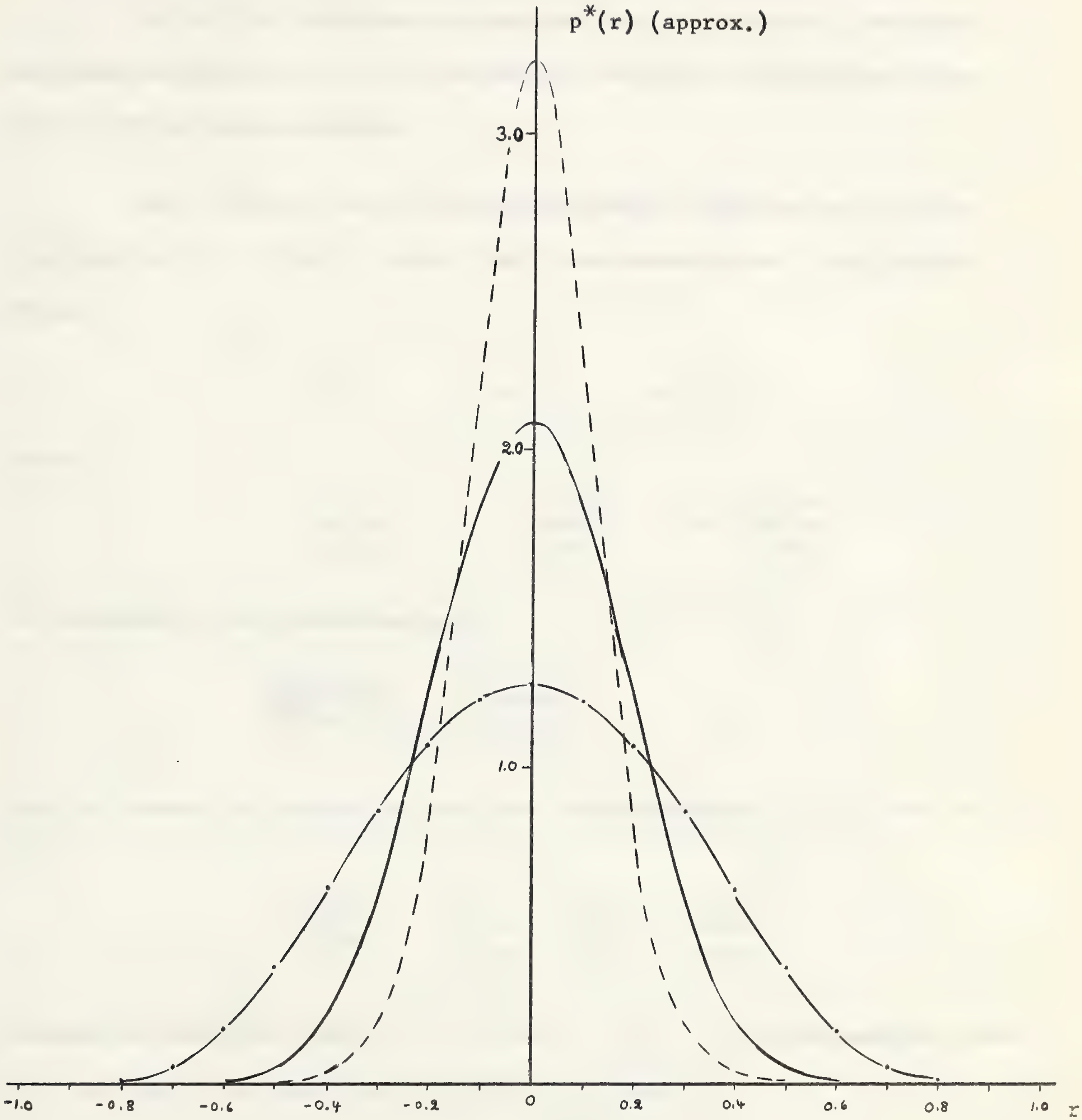

Figure 1. The approximate density function of the product-moment correlation r between two stationary linear Markov series with fitted means for parameter values $\alpha = -0.5, 0.0, 0.5$ and for sample size $n = 30$.

$\alpha = -0.5$ ---, $\alpha = 0.0$ ——, $\alpha = 0.5$ — · — .

APPENDIX I

A METHOD OF INVERSION

It will be convenient to consider here the method of inversion, developed by McGregor [9], for determining the joint distribution of two ratios with a common numerator.

Let $f(C,D,E)$ be the probability density function of the joint distribution of C, D and E, where C and D are non-negative. The transformation

$$r_1 = \frac{E}{C} , \qquad r_2 = \frac{E}{D} , \qquad u = C + D$$

yields

$$C = \frac{r_2 u}{r_1+r_2} , \qquad D = \frac{r_1 u}{r_1+r_2} , \qquad E = \frac{r_1 r_2 u}{r_1+r_2} .$$

The Jacobian of the transformation is

$$\frac{\partial(C,D,E)}{\partial(r_1,r_2,u)} = \frac{r_1 r_2 u^2}{(r_1+r_2)^3} .$$

Thus the probability density of the joint distribution of r_1, r_2 and u is

$$f\left(\frac{r_2 u}{r_1+r_2} , \frac{r_1 u}{r_1+r_2} , \frac{r_1 r_2 u}{r_1+r_2}\right) \frac{r_1 r_2 u^2}{(r_1+r_2)^3} .$$

Integrating this with respect to u from 0 to ∞ gives the marginal density function of the joint distribution of r_1 and r_2,

$$h(r_1,r_2) = \int_0^\infty f\left(\frac{r_2 u}{r_1+r_2} , \frac{r_1 u}{r_1+r_2} , \frac{r_1 r_2 u}{r_1+r_2}\right) \frac{r_1 r_2}{(r_1+r_2)^3} u^2 \, du$$

Let $M(T,S,U) = \mathcal{E}[e^{TC+SD+UE}]$ be the joint moment-generating function of C, D and E. Then

$$M(T,S,U) = \iiint e^{TC+SD+UE} f(C,D,E)\, dC\, dD\, dE \quad .$$

Using the complex Fourier inversion formula (see Tranter [17]) for the moment-generating function we have

$$f(C,D,E) = \frac{1}{(2\pi i)^3} \iiint M(T,S,U) e^{-(TC+SD+UE)}\, dU\, dS\, dT \ ,$$

and

$$f\left(\frac{r_2 u}{r_1+r_2}, \frac{r_1 u}{r_1+r_2}, \frac{r_1 r_2 u}{r_1+r_2}\right)$$

$$= \frac{1}{(2\pi i)^3} \iiint M(T,S,U) \exp\left[-\left(\frac{Tr_2}{r_1+r_2} + \frac{Sr_1}{r_1+r_2} + \frac{Ur_1 r_2}{r_1+r_2}\right) u\right] dU\, dS\, dT \ , \tag{I.1}$$

where the paths of integration are along the imaginary axes in the U, S and T planes or along any allowable deformation of these paths.

Let $$v = \frac{Tr_2}{r_1+r_2} + \frac{Sr_1}{r_1+r_2} + \frac{Ur_1 r_2}{r_1+r_2} \ ,$$

then $$U = \frac{v(r_1+r_2) - Tr_2 - Sr_1}{r_1 r_2} \ , \quad dU = \left(\frac{r_1+r_2}{r_1 r_2}\right) dv$$

and from (I.1)

$$f\left(\frac{r_2 u}{r_1+r_2}, \frac{r_1 u}{r_1+r_2}, \frac{r_1 r_2 u}{r_1+r_2}\right)$$

$$= \frac{1}{(2\pi i)^3} \iiint M\left(T,S, \frac{v(r_1+r_2)-Tr_2-Sr_1}{r_1 r_2}\right) e^{-vu} \left(\frac{r_1+r_2}{r_1 r_2}\right) dv\, dS\, dT \ ,$$

where integration with respect to v is along the transformed path in the v-plane.

Thus

$$\frac{r_1 r_2}{(r_1+r_2)^3} \; f\left(\frac{r_2 u}{r_1+r_2}, \frac{r_1 u}{r_1+r_2}, \frac{r_1 r_2 u}{r_1+r_2}\right)$$

$$= \frac{1}{(2\pi i)^3} \iiint M\left(T, S, \frac{v(r_1+r_2)-Tr_2-Sr_1}{r_1 r_2}\right) \frac{e^{-vu}}{(r_1+r_2)^2} \, dv \, dS \, dT. \quad \text{(I.2)}$$

Multiplying (I.2) by e^{vu} and integrating with respect to u (changing the order of integration in the right member), we obtain

$$\int_0^\infty \frac{r_1 r_2}{(r_1+r_2)^3} \; f\left(\frac{r_2 u}{r_1+r_2}, \frac{r_1 u}{r_1+r_2}, \frac{r_1 r_2 u}{r_1+r_2}\right) e^{vu} \, du$$

$$= \frac{1}{(r_1+r_2)^2} \frac{1}{(2\pi i)^2} \iint \left\{ \int_0^\infty \left[\frac{1}{2\pi i} \int M\left(T,S,\frac{v(r_1+r_2)-Tr_2-Sr_1}{r_1 r_2}\right) e^{-vu} dv \right] \times e^{vu} du \right\} dS \, dT$$

$$= \frac{1}{(r_1+r_2)^2} \frac{1}{(2\pi i)^2} \iint M\left(T,S,\frac{v(r_1+r_2)-Tr_2-Sr_1}{r_1 r_2}\right) dS \, dT \,.$$

Finally, differentiating twice under the integral signs with respect to v and setting $v = 0$, we get

$$h(r_1,r_2) = \int_0^\infty \frac{r_1 r_2}{(r_1+r_2)^3} \; f\left(\frac{r_2 u}{r_1+r_2}, \frac{r_1 u}{r_1+r_2}, \frac{r_1 r_2 u}{r_1+r_2}\right) u^2 \, du$$

$$= \frac{1}{(r_1+r_2)^2} \frac{1}{(2\pi i)^2} \iint \frac{\partial^2}{\partial v^2} \left[M\left(T,S,\frac{v(r_1+r_2)-Tr_2-Sr_1}{r_1 r_2}\right) \right]\Bigg|_{v=0} dS \, dT. \quad \text{(I.3)}$$

APPENDIX II

THE ASYMPTOTIC EVALUATION OF A CLASS OF DETERMINANTS ARISING IN STATISTICS

The following method of evaluation of a certain class of determinants was developed in detail McGregor [8].

In determining the joint moment-generating function of several quadratic forms in normal random variables, a determinant of the following form is frequently encountered:

$$D_n^{(o)} = \left| \begin{array}{c|c|c} \underline{B}^{(o)}_{m,2m} & \underline{O}_{m,n-4m} & \underline{G}_{m,2m} \\ \hline \multicolumn{3}{c}{\underline{C}_{n-2m,n}} \\ \hline \multicolumn{2}{c|}{\underline{O}_{m,n-2m}} & \underline{H}_{m,2m} \end{array} \right| , \qquad (II.1)$$

where

$$\underline{B}^{(j)}_{m,2m} = \begin{bmatrix} b^{(j)}_{11} & b^{(j)}_{12} & \cdots & b^{(j)}_{1,2m} \\ b^{(j)}_{21} & b^{(j)}_{22} & \cdots & b^{(j)}_{2,2m} \\ \cdots & \cdots & \cdots & \cdots \\ b^{(j)}_{m1} & b^{(j)}_{m2} & \cdots & b^{(j)}_{m,2m} \end{bmatrix} ,$$

$\underline{O}_{p,q}$ is the $p \times q$ zero matrix, $\underline{G}_{m,2m}$ and $\underline{H}_{m,2m}$ are $m \times 2m$ matrices with elements $g_{s,t}$ and $h_{s,t}$ respectively, and $\underline{C}_{n-2m,n}$ is the $(n-2m) \times n$ circulant type matrix with first row

$$(1, c_1, c_2, \ldots, c_{m-1}, c_m, c_{m-1}, \ldots, c_2, c_1, 1, 0, 0, \ldots, 0) .$$

Example. For $n = 11$ and $m = 2$, the determinant has the form

$$\begin{vmatrix}
b_{11}^{(o)} & b_{12}^{(o)} & b_{13}^{(o)} & b_{14}^{(o)} & 0 & 0 & 0 & g_{11} & g_{12} & g_{13} & g_{14} \\
b_{21}^{(o)} & b_{22}^{(o)} & b_{23}^{(o)} & b_{24}^{(o)} & 0 & 0 & 0 & g_{21} & g_{22} & g_{23} & g_{24} \\
1 & c_1 & c_2 & c_1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & c_1 & c_2 & c_1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & c_1 & c_2 & c_1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & c_1 & c_2 & c_1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & c_1 & c_2 & c_1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & c_1 & c_2 & c_1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & c_1 & c_2 & c_1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & h_{11} & h_{12} & h_{13} & h_{14} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & h_{21} & h_{22} & h_{23} & h_{24}
\end{vmatrix} .$$

The $n \times n$ determinant $D_n^{(o)}$ may be reduced to one of order $(n-1)$ by multiplying the first column successively by

$c_1, c_2, \ldots, c_{m-1}, c_m, c_{m-1}, \ldots, c_2, c_1$ and 1, subtracting from the second, third, ..., $(2m+1)$th columns respectively and then deleting the first column and $(m+1)$th row. This gives

$$D_n^{(o)} = (-1)^m D_{n-1}^{(1)} \quad , \text{ where,}$$

$$D_{n-1}^{(1)} = \left| \begin{array}{c|c|c} \underline{B}_{m,2m}^{(1)} & \underline{O}_{m,n-1-4m} & \underline{G}_{m,2m} \\ \hline \multicolumn{3}{c}{\underline{C}_{n-1-2m,n-1}} \\ \hline \multicolumn{2}{c|}{\underline{O}_{m,n-1-2m}} & \underline{H}_{m,2m} \end{array} \right| .$$

Repeating this process $(n-2m)$ times, we obtain

$$D_n^{(o)} = (-1)^{m(n-2m)} D_{2m}^{(n-2m)} ,$$

where

$$D_{2m}^{(n-2m)} = \left| \begin{array}{c} \underline{B}_{m,2m}^{(n-2m)} + \underline{G}_{m,2m} \\ \hline \underline{H}_{m,2m} \end{array} \right| . \tag{II.2}$$

At any stage, the elements forming the ith row $(i = 1,2,\ldots,m)$ of $\underline{B}_{m,2m}^{(j)}$ may be obtained from those of the preceding stage by the relation

$$(b_{i1}^{(j)}, b_{i2}^{(j)}, \ldots, b_{i,2m}^{(j)}) = (b_{i1}^{(j-1)}, b_{i2}^{(j-1)}, \ldots, b_{i,2m}^{(j-1)}) \, \underline{T}_{2m} ,$$

where

$$\underline{T}_{2m} = \begin{bmatrix} -c_1 & -c_2 & \ldots & -c_{m-1} & -c_m & -c_{m-1} & \ldots & -c_2 & -c_1 & -1 \\ 1 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 1 & 0 \end{bmatrix} . \tag{II.3}$$

Thus

$$\underline{B}_{m,2m}^{(n-2m)} = \underline{B}_{m,2m}^{(o)} \; \underline{T}_{2m}^{n-2m} . \tag{II.4}$$

Let $p_1, p_2, \ldots, p_{2m}$ be the roots of

$$p^{2m} + c_1 p^{2m-1} + c_2 p^{2m-2} + \ldots + c_{m-1} p^{m+1}$$
$$+ c_m p^m + c_{m-1} p^{m-1} + \ldots + c_2 p^2 + c_1 p + 1 = 0 , \tag{II.5}$$

and let

$$\underline{P}_{2m}^{(k)} = \begin{bmatrix} p_1^{k+2m-1} & p_2^{k+2m-1} & \cdots & p_{2m}^{k+2m-1} \\ p_1^{k+2m-2} & p_2^{k+2m-2} & \cdots & p_{2m}^{k+2m-2} \\ \cdots & \cdots & \cdots & \cdots \\ p_1^{k} & p_2^{k} & \cdots & p_{2m}^{k} \end{bmatrix} .$$

Then from equations (II.3) and (II.5)

$$\underline{T}_{2m}\,\underline{P}_{2m}^{(k)} = \underline{P}_{2m}\,(k+1) \;.$$

Thus

$$\underline{T}_{2m} = \underline{T}_{2m}\,\underline{P}_{2m}(0)\;\;\underline{P}_{2m}^{-1}(0) = \underline{P}_{2m}(1)\;\;\underline{P}_{2m}^{-1}(0) \;,$$

and it follows by induction on k that

$$\underline{T}_{2m}^{k} = \underline{P}_{2m}(k)\;\;\underline{P}_{2m}^{-1}\,(0) \;. \tag{II.6}$$

From equations (II.4) and (II.6) we obtain

$$\underline{B}_{m,2m}^{(n-2m)} = \underline{B}_{m,2m}^{(o)}\;\;\underline{P}_{2m}(n-2m)\;\underline{P}_{2m}^{-1}(0) \;. \tag{II.7}$$

Because of the symmetry of the coefficients in equation (II.5), we may take

$$\left.\begin{aligned} &p_1 = \varphi_1,\; p_3 = \varphi_2,\; \ldots,\; p_{2m-1} = \varphi_m \;,\quad \text{and} \\ &p_2 = 1/\varphi_1,\; p_4 = 1/\varphi_2,\; \ldots,\; p_{2m} = 1/\varphi_m \;. \end{aligned}\right\} \text{(II.8)}$$

In the subsequent discussion, we shall assume that each element of the original determinant $D_n^{(o)}$ is of bounded modulus as $n \to \infty$, that $|\varphi_j| < 1$ $(j = 1,2,\ldots,m)$, that $c_1, c_2, \ldots, c_m$ are independent of n and that m is $O(1)$ with respect to n.

From equations (II.2) and (II.7), we obtain

$$D_{2m}^{(n-2m)} = \left| \frac{\underline{B}_{m,2m}^{(o)}\ \underline{P}_{2m}\ (n-2m)\ \underline{P}_{2m}^{-1}(0) + \underline{G}_{m,2m}}{\underline{H}_{m,2m}} \right| . \qquad \text{(II.9)}$$

Let

$$\underline{P}_{2m}^{-1}(0) = \underline{\prod} = \begin{bmatrix} \pi_{11} & \pi_{12} & \cdots & \pi_{1,2m} \\ \pi_{21} & \pi_{22} & \cdots & \pi_{2,2m} \\ \cdots & \cdots & \cdots & \cdots \\ \pi_{2m,1} & \pi_{2m,2} & \cdots & \pi_{2m,2m} \end{bmatrix} . \qquad \text{(II.10)}$$

A typical element of the upper partition

$\underline{B}_{m,2m}^{(o)}\ \underline{P}_{2m}\ (n-2m)\ \underline{P}_{2m}^{-1}(0) + \underline{G}_{m,2m}$, of $D_{2m}^{(n-2m)}$, is

$$e_{st} = g_{st} + \sum_{i=1}^{2m} \sum_{j=1}^{2m} b_{sj}^{(o)}\ p_i^{n-j}\ \pi_{it}$$

$$= g_{st} + \sum_{i=1}^{m} \sum_{j=1}^{2m} b_{sj}^{(o)}\ p_{2i}^{n-j}\ \pi_{2i,t} + \sum_{i=1}^{m} \sum_{j=1}^{2m} b_{sj}^{(o)}\ p_{2i-1}^{n-j}\ \pi_{2i-1,t}$$

$$\sim \sum_{i=1}^{m} \sum_{j=1}^{2m} b_{sj}^{(o)}\ p_{2i}^{n-j}\ \pi_{2i,t} ,$$

where the terms omitted are relatively $O(\epsilon^n)$ for some $|\epsilon| < 1$. Hence we may write

$$\underline{B}^{(o)}_{m,2m}\ \underline{P}_{2m}(n-2m)\ \underline{P}^{-1}_{2m}(0) + \underline{G}_{m,2m}$$

$$\sim \underline{B}^{(o)}_{m,2m} \begin{bmatrix} p_2^{n-1} & p_4^{n-1} & \cdots & p_{2m}^{n-1} \\ p_2^{n-2} & p_4^{n-2} & \cdots & p_{2m}^{n-2} \\ \cdots & \cdots & \cdots & \cdots \\ p_2^{n-2m} & p_4^{n-2m} & \cdots & p_{2m}^{n-2m} \end{bmatrix} \begin{bmatrix} \pi_{21} & \pi_{22} & \cdots & \pi_{2,2m} \\ \pi_{41} & \pi_{42} & \cdots & \pi_{4,2m} \\ \cdots & \cdots & \cdots & \cdots \\ \pi_{2m,1} & \pi_{2m,2} & \cdots & \pi_{2m,2m} \end{bmatrix}. \quad \text{(II.11)}$$

For the remaining discussion, there will be no ambiguity in writing $\underline{B}^{(o)}_{m,2m}$ simply as $\underline{B}$, and $\underline{H}_{m,2m}$ as $\underline{H}$. From equations (II.9) and (II.11), we obtain

$$D_{2m}^{(n-2m)}$$

$$\sim \left| \begin{array}{c} \underline{B} \begin{bmatrix} p_2^{n-1} & p_4^{n-1} & \cdots & p_{2m}^{n-1} \\ p_2^{n-2} & p_4^{n-2} & \cdots & p_{2m}^{n-2} \\ \cdots & \cdots & \cdots & \cdots \\ p_2^{n-2m} & p_4^{n-2m} & \cdots & p_{2m}^{n-2m} \end{bmatrix} \begin{bmatrix} \pi_{21} & \pi_{22} & \cdots & \pi_{2,2m} \\ \pi_{41} & \pi_{42} & \cdots & \pi_{4,2m} \\ \cdots & \cdots & \cdots & \cdots \\ \pi_{2m,1} & \pi_{2m,2} & \cdots & \pi_{2m,2m} \end{bmatrix} \\ \hline \underline{H} \end{array} \right| . \quad \text{(II.12)}$$

The determinant of the right of equation (II.12) may be written as the product of the two determinants

$$K_1^{(1)} = \left| \begin{array}{c|c} \underline{B} \begin{bmatrix} p_2^{n-1} & p_4^{n-1} & \cdots & p_{2m}^{n-1} \\ p_2^{n-2} & p_4^{n-2} & \cdots & p_{2m}^{n-2} \\ \cdots & \cdots & \cdots & \cdots \\ p_2^{n-2m} & p_4^{n-2m} & \cdots & p_{2m}^{n-2m} \end{bmatrix} & \underline{O}_m \\ \hline \underline{O}_m & \underline{I}_m \end{array} \right| ,$$

where $\underline{0}_m$ and $\underline{I}_m$ are respectively the $m \times m$ zero and identity matrices, and

$$K_2 = \left| \underbrace{\begin{bmatrix} \pi_{21} & \pi_{22} & \cdots & \pi_{2,2m} \\ \pi_{41} & \pi_{42} & \cdots & \pi_{4,2m} \\ \cdot & \cdot & \cdot\;\cdot\;\cdot & \cdot \\ \pi_{2m,1} & \pi_{2m,2} & \cdots & \pi_{2m,2m} \end{bmatrix}}_{\underline{H}} \right| . \tag{II.13}$$

$K_1^{(1)}$ may be collapsed along the diagonal of $\underline{I}_m$ giving

$$K_1^{(1)} = \left| \quad \underline{B} \quad \begin{bmatrix} p_2^{n-1} & p_4^{n-1} & \cdots & p_{2m}^{n-1} \\ p_2^{n-2} & p_4^{n-2} & \cdots & p_{2m}^{n-2} \\ \cdot & \cdot & \cdot\;\cdot\;\cdot & \cdot \\ p_2^{n-2m} & p_4^{n-2m} & \cdots & p_{2m}^{n-2m} \end{bmatrix} \right| .$$

If the indicated matrix multiplication is performed, $(p_2\, p_4 \cdots p_{2m})^{n-2m}$ may be factored from the determinant giving

$$K_1^{(1)} = (p_2 p_4 \cdots p_{2m})^{n-2m} \left| \quad \underline{B} \quad \begin{bmatrix} p_2^{2m-1} & p_4^{2m-1} & \cdots & p_{2m}^{2m-1} \\ p_2^{2m-2} & p_4^{2m-2} & \cdots & p_{2m}^{2m-2} \\ \cdot & \cdot & \cdot\;\cdot\;\cdot & \cdot \\ 1 & 1 & \cdots & 1 \end{bmatrix} \right|$$

$$= (p_2 p_4 \cdots p_{2m})^{n-2m}$$

continued

$$\times \left|\begin{array}{c|c} \underline{B}\begin{bmatrix} p_2^{2m-1} & p_4^{2m-1} & \cdots & p_{2m}^{2m-1} \\ p_2^{2m-2} & p_4^{2m-2} & \cdots & p_{2m}^{2m-2} \\ \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & \cdots & 1 \end{bmatrix} & \underline{B}\begin{bmatrix} p_1^{2m-1} & p_3^{2m-1} & \cdots & p_{2m-1}^{2m-1} \\ p_1^{2m-2} & p_3^{2m-2} & \cdots & p_{2m-1}^{2m-2} \\ \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & \cdots & 1 \end{bmatrix} \\ \hline \underline{O}_m & \underline{I}_m \end{array}\right|,$$

which may be written as,

$$K_1^{(1)} = (p_2 p_4 \cdots p_{2m})^{n-2m} \left|\begin{array}{c} \underline{B} \\ \hline \begin{bmatrix} \pi_{11} & \pi_{12} & \cdots & \pi_{1,2m} \\ \pi_{31} & \pi_{32} & \cdots & \pi_{3,2m} \\ \cdots & \cdots & \cdots & \cdots \\ \pi_{2m-1,1} & \pi_{2m-1,2} & \cdots & \pi_{2m-1,2m} \end{bmatrix} \end{array}\right|$$

$$\times \begin{vmatrix} p_2^{2m-1} & p_4^{2m-1} & \cdots & p_{2m}^{2m-1} & p_1^{2m-1} & p_3^{2m-1} & \cdots & p_{2m-1}^{2m-1} \\ p_2^{2m-2} & p_4^{2m-2} & \cdots & p_{2m}^{2m-2} & p_1^{2m-2} & p_3^{2m-2} & \cdots & p_{2m-1}^{2m-2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ p_2 & p_4 & \cdots & p_{2m} & p_1 & p_3 & \cdots & p_{2m-1} \\ 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 \end{vmatrix} .$$

It can be shown that

$$\begin{vmatrix} p_2^{2m-1} & p_4^{2m-1} & \cdots & p_{2m}^{2m-1} & p_1^{2m-1} & p_3^{2m-1} & \cdots & p_{2m-1}^{2m-1} \\ p_2^{2m-2} & p_4^{2m-2} & \cdots & p_{2m}^{2m-2} & p_1^{2m-2} & p_3^{2m-2} & \cdots & p_{2m-1}^{2m-2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ p_2 & p_4 & \cdots & p_{2m} & p_1 & p_3 & \cdots & p_{2m-1} \\ 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 \end{vmatrix}$$

continued

$$= (-1)^{m(m+1)/2} \prod_{1 \leq i < j \leq 2m} (p_i - p_j) .$$

(see for example Aitken [1]). Thus

$$K_1^{(1)} = (p_2 p_4 \cdots p_{2m})^{n-2m} K_1 (-1)^{m(m+1)/2} \prod_{1 \leq i < j \leq 2m} (p_i - p_j) ,$$

where

$$K_1 = \left| \begin{array}{c} \underline{B} \\ \hline \begin{bmatrix} \pi_{11} & \pi_{12} & \cdots & \pi_{1,2m} \\ \pi_{31} & \pi_{32} & \cdots & \pi_{3,2m} \\ \cdot & \cdot & \cdot & \cdot \\ \pi_{2m-1,1} & \pi_{2m-1,2} & \cdots & \pi_{2m-1,2m} \end{bmatrix} \end{array} \right| . \qquad \text{(II.14)}$$

Then, from equation (II.12), we obtain

$$D_{2m}^{(n-2m)} \sim (p_2 p_4 \cdots p_{2m})^{n-2m} K_1 K_2 (-1)^{m(m+1)/2} \prod_{1 \leq i < j \leq 2m} (p_i - p_j) .$$

(II.15)

Consider the matrix $\underline{\prod} = (\pi_{ij})$ defined by equation (II.10). Let $N(s;t)$ denote the sum of all distinct products of $p_1, p_2, \ldots, p_{2m}$ taken s at a time and excluding p_t , $(t = 1,2,\ldots,2m ;\ s = 1,2,\ldots,2m-1)$. Let

$$M(t) = \prod_{i=1}^{2m}{}' (p_t - p_i) \qquad (t = 1,2,\ldots,2m) , \qquad \text{(II.16)}$$

where $\prod'$ indicates that $i \neq t$. In terms of these quantities, it can be shown that

$$\pi_{ij} = (-1)^{j-1} \frac{N(j-1\,;i)}{M(i)} \qquad (j = 1,2,3,\ldots,2m), \qquad \text{(II.17)}$$

where we define $N(0;i) = 1$ $(i = 1,2,\ldots,2m)$. (See for example Aitken

[1]). From equations (II.14), (II.16) and (II.17), we may write

$$K_1 = \left[\prod_{t=1}^{m} M(2t-1) \right]^{-1}$$

$$\times \left| \frac{\underline{B}}{\begin{bmatrix} 1 & -N(1;1) & N(2;1) & \dots & -N(2m-1;1) \\ 1 & -N(1;3) & N(2;3) & \dots & -N(2m-1;3) \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & -N(1;2m-1) & N(2;2m-1) & \dots & -N(2m-1;2m-1) \end{bmatrix}} \right| . \qquad \text{(II.18)}$$

More generally, let $N(s;t_1,t_2,\dots,t_k)$, $(s \leq 2m-k)$, denote the sum of all distinct products of s elements chosen from the subset of $p_1,p_2,\dots,p_{2m}$ which excludes p_{t_1}, p_{t_2}, ..., p_{t_k}. For $s > 2m-k$ define $N(s;t_1, t_2,\dots, t_k)$ to be 0. Then

$$\left.\begin{aligned} N(s;1) &= p_k\, N(s-1;1,k) + N(s;1,k) , \\ N(s;k) &= p_1\, N(s-1;1,k) + N(s;1,k) , \\ N(s;k) - N(s;1) &= (p_1 - p_k)\, N(s-1;1,k) , \end{aligned}\right\} \qquad \text{(II.19)}$$

for $k = 3,5,\dots,2m-1$ and $s = 1,2,\dots,2m-1$. Similarly,

$$\begin{aligned} &N(s-j;1,3,\dots,2j-1,k) - N(s-j;1,3,\dots,2j+1) \\ &\quad = (p_{2j+1} - p_k)\, N(s-j);1,3,\dots,2j+1,k) , \end{aligned} \qquad \text{(II.20)}$$

for $k = 2j+3,\ 2j+5,\ \dots,\ 2m-1$; $s = j+1,\ j+2,\ \dots,\ 2m-1$ and $j = 1,2,\dots,m$.

Let $\underline{Q}(\tau)$ be the $2m \times 2m$ matrix

$$\underline{Q}(\tau) = \left[\begin{array}{c|c} \underline{I}_{m+\tau,m+\tau} & \underline{0}_{m+\tau,m-\tau} \\ \hline \underline{0}_{m-\tau,m+\tau} & \underline{V}_{m-\tau,m-\tau} \end{array} \right] \qquad \text{(II.21)}$$

$$(\tau = 0,1,2,\dots,m-2) ,$$

where $\underline{V}_{m-\tau,m-\tau}$ is the $(m-\tau) \times (m-\tau)$ matrix

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 1 & 0 & \cdots & 0 & 0 \\ -1 & 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -1 & 0 & 0 & \cdots & 1 & 0 \\ -1 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix} .$$

Then $\left|\underline{Q}(\tau)\right| = 1 \quad (\tau = 0,1,2,\ldots,m-2)$.

From equations (II.18), (II.19), (II.20) and (II.21) we obtain

$$K_1 = \left[\prod_{t=1}^{m} M(2t-1)\right]^{-1}$$

$$\times \left| \underline{Q}(0) \begin{bmatrix} & & \underline{B} & & \\ \hline 1 & -N(1;1) & N(2;1) & \cdots & -N(2m-1;1) \\ 1 & -N(1;3) & N(2;3) & \cdots & -N(2m-1;3) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & -N(1;2m-1) & N(2;2m-1) & \cdots & -N(2m-1;2m-1) \end{bmatrix} \right|$$

$$= \left[\prod_{t=1}^{m} M(2t-1)\right]^{-1}$$

$$\times \left| \begin{matrix} \underline{B} \\ \hline \begin{bmatrix} 1 & -N(1;1) & N(2;1) & \cdots & -N(2m-1;1) \\ 0 & (p_3-p_1) & -(p_3-p_1)N(1;1,3) & \cdots & (p_3-p_1)N(2m-2;1,3) \\ 0 & (p_5-p_1) & -(p_5-p_1)N(1;1,5) & \cdots & (p_5-p_1)N(2m-2;1,5) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & (p_{2m-1}-p_1) & -(p_{2m-1}-p_1)N(1;1,2m-1) & \cdots & (p_{2m-1}-p_1)N(2m-2;1,2m-1) \end{bmatrix} \end{matrix} \right|$$

continued

$$= \frac{\prod_{i=1}^{m-1} (p_{2i+1} - p_1)}{\prod_{t=1}^{m} M(2t-1)} \left| \begin{array}{c} \underline{B} \\ \hline \begin{bmatrix} 1 & -N(1;1) & N(2;1) & \dots & -N(2m-1;1) \\ 0 & 1 & -N(1;1,3) & \dots & N(2m-2;1,3) \\ 0 & 1 & -N(1;1,5) & \dots & N(2m-2;1,5) \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 1 & -N(1;1,2m-1) & \dots & N(2m-2;1,2m-1) \end{bmatrix} \end{array} \right| .$$

Repeating this process using $\underline{Q}(1), \underline{Q}(2), \dots, \underline{Q}(m-2)$ successively, we ultimately obtain

$$K_1 = \frac{\prod_{1 \le i < j \le m} (p_{2j-1} - p_{2i-1})}{\prod_{t=1}^{m} M(2t-1)} \left| \frac{\underline{B}}{\underline{F}} \right| , \qquad \text{(II.22)}$$

where $\underline{F} = (f_{ij})$ is the $m \times 2m$ matrix with elements

$$f_{ij} = 0, \quad i > j ,$$

$$f_{ii} = 1,$$

$$f_{ij} = (-1)^{i+j} N(j-i;1,3,\dots,2i-1) , \quad i < j .$$

Thus $\left| \frac{\underline{B}}{\underline{F}} \right| =$

$$\left| \begin{array}{c} \underline{B} \\ \hline \begin{bmatrix} 1 & -N(1;1) & N(2;1) & \dots & N(2m-2;1) & N(2m-1;1) \\ 0 & 1 & -N(1;1,3) & \dots & -N(2m-3;1,3) & N(2m-2;1,3) \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \dots & (-1)^{m-1}N(m-1;1,3,\dots,2m-1) & (-1)^{m}N(m;1,3,\dots,2m-1) \end{bmatrix} \end{array} \right| .$$

Let $\underline{Q}^*(\tau)$ be the $2m \times 2m$ matrix

$$\underline{Q}^*(\tau) = \left[\begin{array}{c|c|c} \underline{I}_{m,m} & \multicolumn{2}{c}{\underline{O}_{m,m}} \\ \hline \underline{O}_{m,m} & \underline{U}_{m-\tau,m-\tau+1} & \underline{O}_{m-\tau,\tau-1} \\ \cline{2-3} & \underline{O}_{\tau,m-\tau} & \underline{I}_{\tau,\tau} \end{array}\right] \qquad (\tau = 1,2,\ldots,m-1),$$

where

$$\underline{U}_{m-\tau,m-\tau+1} = \begin{bmatrix} 1 & p_{2\tau+1} & & & \\ & 1 & p_{2\tau+3} & & (0) \\ & & \ddots & \ddots & \\ (0) & & & 1 & p_{2m-1} \end{bmatrix} .$$

Then $|\underline{Q}^*(\tau)| = 1$ $(\tau = 1,2,\ldots,m-1)$. Successive premultiplication of $\left[\frac{\underline{B}}{\underline{F}}\right]$ by $\underline{Q}^*(1)$, $\underline{Q}^*(2)$, ..., $\underline{Q}^*(m-1)$ yields

$$\begin{aligned} \left|\frac{\underline{B}}{\underline{F}}\right| &= \left|\underline{Q}^*(m-1)\,\underline{Q}^*(m-2)\,\ldots\,\underline{Q}^*(2)\,\underline{Q}^*(1)\left[\frac{\underline{B}}{\underline{F}}\right]\right| \\ &= \left|\frac{\underline{B}}{\underline{F}^*}\right| \quad , \end{aligned} \qquad \text{(II.23)}$$

where $\underline{F}^*$ is the $m \times 2m$ circulant type matrix with first row $\{1, -N(1;1,3,\ldots,2m-1), N(2;1,3,\ldots,2m-1), \ldots, (-1)^{m-1} N(m-1;1,3,\ldots,2m-1), (-1)^m N(m;1,3,\ldots,2m-1), 0, 0, \ldots, 0\}$.

By definition,

$$N(j;1,3,\ldots,2m-1) = \sum_{1 \le i_1 < i_2 < \ldots < i_j \le m} \cdots \sum p_{2i_1} p_{2i_2} \cdots p_{2i_j} \qquad (j = 1,2,\ldots,m) .$$

Using relations (II.8) this becomes

$$N(j;1,3,\ldots,2m-1) = \frac{1}{\varphi_1\varphi_2\cdots\varphi_m} \sum_{1 \le k_1 < k_2 < \ldots < k_{m-j} \le m} \cdots \sum \varphi_{k_1}\varphi_{k_2}\cdots\varphi_{k_{m-j}} \qquad (j = 1,2,\ldots,m-1) ,$$

while

$$N(m;1,3,\ldots,2m-1) = \frac{1}{\varphi_1\varphi_2\cdots\varphi_m} \quad .$$

Let

$$a_j = (-1)^j \sum_{1\le k_1<k_2\ldots<k_j\le m} \cdots \sum \varphi_{k_1}\varphi_{k_2}\cdots\varphi_{k_j} \qquad \text{(II.24)}$$

$$(j = 1,2,\ldots,m) \quad .$$

Then $\varphi_1, \varphi_2, \ldots, \varphi_m$ are the roots of

$$\varphi^m + a_1\varphi^{m-1} + a_2\varphi^{m-2} + \ldots + a_{m-1}\varphi + a_m = 0 \,, \qquad \text{(II.25)}$$

and

$$N(j;1,3,\ldots,2m-1) = (-1)^j \frac{a_{m-j}}{a_m} \,, \qquad \text{(II.26)}$$

where we have taken $a_o \equiv 1$. Furthermore, by equations (II.5), (II.8) and (II.25),

$$\begin{aligned} &\varphi^{2m} + c_1\varphi^{2m-1} + c_2\varphi^{2m-2} + \ldots + c_{m-1}\varphi^{m+1} + c_m\varphi^m \\ &\quad + c_{m-1}\varphi^{m-1} + \ldots + c_2\varphi^2 + c_1\varphi + 1 \\ &\quad = (\varphi-\varphi_1)(\varphi-\varphi_2)\ \ldots\ (\varphi-\varphi_m)(\varphi - \frac{1}{\varphi_1})(\varphi - \frac{1}{\varphi_2})\ \ldots\ (\varphi - \frac{1}{\varphi_m}) \\ &\quad = \frac{1}{a_m}(\varphi^m + a_1\varphi^{m-1} + \ldots + a_{m-1}\varphi + a_m) \\ &\qquad\quad \times(a_m\varphi^m + a_{m-1}\varphi^{m-1} + \ldots + a_1\varphi + 1) \,, \end{aligned}$$

and, by identifying coefficients of φ^j, we obtain

$$c_j = \frac{1}{a_m}(a_{m-j} + a_1 a_{m-j+1} + \ldots + a_j a_m) \qquad \text{(II.27)}$$

$$(j = 1,2,\ldots,m) \quad .$$

Thus $\underline{F}^*$ becomes the $m \times 2m$ circulant type matrix with first row

$$\{1, \frac{a_{m-1}}{a_m}, \frac{a_{m-2}}{a_m}, \ldots, \frac{a_2}{a_m}, \frac{a_1}{a_m}, \frac{1}{a_m}, 0, 0, \ldots, 0\} \quad .$$

From equation (II.23),

$$\left| \frac{\underline{B}}{\underline{F}} \right| = a_m^{-m} \left| \begin{array}{c} \underline{B} \\ \hline \begin{bmatrix} a_m & a_{m-1} & \cdots & a_1 & 1 & & & (0) \\ & a_m & a_{m-1} & \cdots & a_1 & 1 & & \\ & & \ddots & \ddots & & \ddots & \ddots & \\ (0) & & & a_m & a_{m-1} & \cdots & a_1 & 1 \end{bmatrix} \end{array} \right| ,$$

and from equation (II.22),

$$K_1 = a_m^{-m} \prod_{1 \leq i < j \leq m} (p_{2j-1} - p_{2i-1}) \left[\prod_{t=1}^{m} M(2t-1) \right]^{-1}$$

$$\times \left| \begin{array}{c} \underline{B} \\ \hline \begin{bmatrix} a_m & a_{m-1} & \cdots & a_1 & 1 & & & (0) \\ & a_m & a_{m-1} & \cdots & a_1 & 1 & & \\ & & \ddots & \ddots & & \ddots & \ddots & \\ (0) & & & a_m & a_{m-1} & \cdots & a_1 & 1 \end{bmatrix} \end{array} \right| . \quad \text{(II.28)}$$

A similar reduction gives

$$K_2 = \prod_{1 \leq i < j \leq m} (p_{2i} - p_{2j}) \left[\prod_{t=1}^{m} M(2t) \right]^{-1}$$

$$\times \left| \begin{array}{c} \begin{bmatrix} 1 & a_1 & \cdots & a_{m-1} & a_m & & & (0) \\ & 1 & a_1 & \cdots & a_{m-1} & a_m & & \\ & & \ddots & \ddots & & \ddots & \ddots & \\ (0) & & & 1 & a_1 & \cdots & a_{m-1} & a_m \end{bmatrix} \\ \hline \underline{H} \end{array} \right| . \quad \text{(II.29)}$$

Using equations (II.8) and (II.24) it may be verified that the following relations hold:

$$(p_2 p_4 \dots p_{2m})^{n-2m} = (-1)^{m(n-2m)} \; a_m^{-(n-2m)} \; ,$$

$$(-1)^{\frac{1}{2}m(m+1)} \prod_{1\le i<j\le 2m} (p_i - p_j)$$

$$= \frac{(-1)^{m+\frac{1}{2}m(m-1)}}{a_m^{2m-1}} \prod_{i,j=1}^{m} (1-\varphi_i\varphi_j) \prod_{1\le i<j\le m} (\varphi_i - \varphi_j)^2 \; ,$$

$$\prod_{t=1}^{2m} M(t) = \frac{(-1)^m}{a_m^{2(2m-1)}} \prod_{i,j=1}^{m} (1-\varphi_i\varphi_j)^2 \prod_{1\le i<j\le m} (\varphi_i - \varphi_j)^4 \; ,$$

$$\prod_{1\le i<j\le m} (p_{2j-1} - p_{2i-1}) = \prod_{1\le i<j\le m} (\varphi_j - \varphi_i) \; ,$$

$$\prod_{1\le i<j\le m} (p_{2j} - p_{2i}) = \frac{(-1)^{\frac{1}{2}m(m-1)}}{a_m^{m-1}} \prod_{1\le i<j\le m} (\varphi_j - \varphi_i) \; .$$

Finally, using these relations and equations (II.2), (II.15), (II.28) and (II.29), we obtain after simplification,

$$D_n^{(o)} \sim a_m^{-(n-2m)} \prod_{i,j=1}^{m} (1-\varphi_i\varphi_j)^{-1}$$

$$\times \left| \underset{\underline{B}}{\begin{bmatrix} a_m & a_{m-1} & \cdots & a_1 & 1 & & & (0) \\ & a_m & a_{m-1} & \cdots & a_1 & 1 & & \\ & & \ddots & & & \ddots & \ddots & \\ (0) & & & a_m & a_{m-1} & \cdots & a_1 & 1 \end{bmatrix}} \right| \left| \underset{\underline{J}_m \underline{H} \underline{J}_{2m}}{\begin{bmatrix} a_m & a_{m-1} & \cdots & a_1 & 1 & & & (0) \\ & a_m & a_{m-1} & \cdots & a_1 & 1 & & \\ & & \ddots & & & \ddots & \ddots & \\ (0) & & & a_m & a_{m-1} & \cdots & a_1 & 1 \end{bmatrix}} \right| , \qquad \text{(II.30)}$$

where $\underline{J}_k$ is the $k \times k$ matrix with units in the secondary diagonal and zeros elsewhere.

APPENDIX III

THE ASYMPTOTIC EVALUATION OF TWO DETERMINANTS ENCOUNTERED IN THIS THESIS

A.) Known Means

In order to obtain the joint moment-generating function of C, D and E (2.6) the determinant of $\underline{A}$ has to be evaluated. $\underline{A}$ is the non-singular, $2n \times 2n$ partitioned matrix,

$$\underline{A} = \left[\begin{array}{c|c} \underline{Q}_1 & \underline{Q}_3 \\ \hline \underline{Q}_3 & \underline{Q}_2 \end{array}\right] , \qquad \text{(III.1)}$$

with the $n \times n$ submatrices

$$\underline{Q}_1 = \begin{bmatrix} 1-2T & -\rho_1 & 0 & \dots & 0 & 0 \\ -\rho_1 & 1+\rho_1^2-2T & -\rho_1 & \dots & 0 & 0 \\ 0 & -\rho_1 & 1+\rho_1^2-2T & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1+\rho_1^2-2T & -\rho_1 \\ 0 & 0 & 0 & \dots & -\rho_1 & 1-2T \end{bmatrix} , \qquad \text{(III.2)}$$

$$\underline{Q}_2 = \begin{bmatrix} 1-2S & -\rho_2 & 0 & \dots & 0 & 0 \\ -\rho_2 & 1+\rho_2^2-2S & -\rho_2 & \dots & 0 & 0 \\ 0 & -\rho_2 & 1+\rho_2^2-2S & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1+\rho_2^2-2S & -\rho_2 \\ 0 & 0 & 0 & \dots & -\rho_2 & 1-2S \end{bmatrix} , \qquad \text{(III.3)}$$

and $\underline{Q}_3$ being a scalar matrix with diagonal elements $(-U)$. Since the matrices are nonsingular, we may write (III.1) as

$$\left[\begin{array}{c|c} \underline{Q}_1 & \underline{Q}_3 \\ \hline \underline{Q}_3 & \underline{Q}_2 \end{array}\right] = \left[\begin{array}{c|c} \underline{Q}_1 & \underline{0} \\ \hline \underline{0} & \underline{Q}_2 \end{array}\right] + \left[\begin{array}{c|c} \underline{0} & \underline{Q}_3 \\ \hline \underline{Q}_3 & \underline{0} \end{array}\right]$$

$$= \left[\begin{array}{c|c} \underline{Q}_1 & \underline{0} \\ \hline \underline{0} & \underline{Q}_2 \end{array}\right] \left[\left(\begin{array}{c|c} \underline{I} & \underline{0} \\ \hline \underline{0} & \underline{I} \end{array}\right) + \left(\begin{array}{c|c} \underline{Q}_1^{-1} & \underline{0} \\ \hline \underline{0} & \underline{Q}_2^{-1} \end{array}\right) \left(\begin{array}{c|c} \underline{0} & \underline{Q}_3 \\ \hline \underline{Q}_3 & \underline{0} \end{array}\right) \right]$$

$$= \left[\begin{array}{c|c} \underline{Q}_1 & \underline{0} \\ \hline \underline{0} & \underline{Q}_2 \end{array}\right] \left[\begin{array}{c|c} \underline{I} & \underline{Q}_1^{-1}\,\underline{Q}_3 \\ \hline \underline{Q}_2^{-1}\underline{Q}_3 & \underline{I} \end{array}\right] . \qquad \text{(III.4)}$$

Thus from (III.1) and (III.4) the determinant of $\underline{A}$ is

$$|\underline{A}| = \left|\begin{array}{c|c} \underline{Q}_1 & \underline{0} \\ \hline \underline{0} & \underline{Q}_2 \end{array}\right| \left|\begin{array}{c|c} \underline{I} & \underline{Q}_1^{-1}\,\underline{Q}_3 \\ \hline \underline{Q}_2^{-1}\,\underline{Q}_3 & \underline{I} \end{array}\right|$$

$$= |\underline{Q}_1|\,|\underline{Q}_2| \left|\begin{array}{c|c} \underline{I} & \underline{Q}_1^{-1}\,\underline{Q}_3 \\ \hline \underline{Q}_2^{-1}\,\underline{Q}_3 & \underline{I} \end{array}\right| . \qquad \text{(III.5)}$$

Again employing the theory of partitioned matrices, Aitken [1], consider

$$\left|\begin{array}{c|c} \underline{I} & \underline{Q}_1^{-1}\,\underline{Q}_3 \\ \hline \underline{Q}_2^{-1}\,\underline{Q}_3 & \underline{I} \end{array}\right| = \frac{1}{|\underline{Q}_2^{-1}\,\underline{Q}_3|} \left|\begin{array}{c|c} \underline{I} & \underline{Q}_1^{-1}\,\underline{Q}_3 \\ \hline \underline{Q}_2^{-1}\,\underline{Q}_3 & \underline{I} \end{array}\right| \left|\begin{array}{c|c} \underline{I} & \underline{0} \\ \hline \underline{0} & \underline{Q}_2^{-1}\,\underline{Q}_3 \end{array}\right|$$

$$= \frac{1}{|\underline{Q}_2^{-1}\,\underline{Q}_3|} \left|\begin{array}{c|c} \underline{I} & \underline{Q}_1^{-1}\,\underline{Q}_3\,\underline{Q}_2^{-1}\,\underline{Q}_3 \\ \hline \underline{Q}_2^{-1}\,\underline{Q}_3 & \underline{Q}_2^{-1}\,\underline{Q}_3 \end{array}\right|$$

$$= \frac{1}{|\underline{Q}_2^{-1}\,\underline{Q}_3|} \left|\begin{array}{c|c} \underline{I}-\underline{Q}_1^{-1}\underline{Q}_3\underline{Q}_2^{-1}\underline{Q}_3 & \underline{Q}_1^{-1}\underline{Q}_3\underline{Q}_2^{-1}\underline{Q}_3 \\ \hline \underline{0} & \underline{Q}_2^{-1}\,\underline{Q}_3 \end{array}\right|$$

$$= |\underline{I} - \underline{Q}_1^{-1}\,\underline{Q}_3\,\underline{Q}_2^{-1}\,\underline{Q}_3| \ , \qquad \text{(III.6)}$$

where $\underline{Q}_3$ is a scalar matrix.

Substituting (III.6) in (III.5), we get

$$|\underline{A}| = |\underline{Q}_1||\underline{Q}_2||\underline{I} - \underline{Q}_1^{-1}\,\underline{Q}_2^{-1}\,\underline{Q}_3^2|$$

$$= |\underline{Q}_1\,\underline{Q}_2 - \underline{Q}_3^2| \;. \qquad \text{(III.7)}$$

Thus using the notation of Appendix II, we may write $|\underline{A}|$ as

$$|\underline{A}| = \rho_1^n \rho_2^n \begin{vmatrix} \beta_{11} & \beta_{12} & \beta_{13} & \beta_{14} & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 \\ \beta_{21} & \beta_{22} & \beta_{23} & \beta_{24} & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 \\ 1 & x_1 & x_2 & x_1 & 1 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & x_1 & x_2 & x_1 & 1 & \ldots & 0 & 0 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 & 0 & \ldots & 1 & x_1 & x_2 & x_1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & \beta_{24} & \beta_{23} & \beta_{22} & \beta_{21} \\ 0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & \beta_{14} & \beta_{13} & \beta_{12} & \beta_{11} \end{vmatrix} , \qquad \text{(III.8)}$$

where

$$\left.\begin{aligned} \beta_{11} &= \frac{(1-2T)(1-2S)}{\rho_1\,\rho_2} + 1 - \frac{U^2}{\rho_1\rho_2} , \\ \beta_{12} &= \frac{-(1-2T)}{\rho_1} - \frac{(1+\rho_2^2-2S)}{\rho_2} , \\ \beta_{21} &= \frac{-(1-2S)}{\rho_2} - \frac{(1+\rho_1^2-2T)}{\rho_1} , \\ x_1 &= \frac{-(1+\rho_1^2-2T)}{\rho_1} - \frac{(1+\rho_2^2-2S)}{\rho_2} , \\ x_2 &= 2 + \frac{(1+\rho_1^2-2T)(1+\rho_2^2-2S)}{\rho_1\rho_2} - \frac{U^2}{\rho_1\rho_2} , \\ \beta_{13} &= \beta_{24} = 1, \quad \beta_{14} = 0, \quad \beta_{22} = x_2 \quad \text{and} \quad \beta_{23} = x_1 \;. \end{aligned}\right\} \qquad \text{(III.9)}$$

(Note; with regard to the notation used in Appendix II, m = 2 in (III.8)).

Following the method of Appendix II, we may ultimately reduce $|\underline{A}|$ to the 4×4 determinant,

$$|\underline{A}| = \rho_1^n \rho_2^n \left| \begin{array}{cc} \underline{B} & \underline{P}_4(n-4)\, \underline{P}_4^{-1}(0) \\ \hline & \underline{H} \end{array} \right| , \qquad \text{(III.10)}$$

where

$$\underline{B} = \begin{pmatrix} \beta_{11} & \beta_{12} & \beta_{13} & \beta_{14} \\ \beta_{21} & \beta_{22} & \beta_{23} & \beta_{24} \end{pmatrix} ,$$

$$\underline{H} = \begin{pmatrix} \beta_{24} & \beta_{23} & \beta_{22} & \beta_{21} \\ \beta_{14} & \beta_{13} & \beta_{12} & \beta_{11} \end{pmatrix} ,$$

$$\underline{P}_4(n-4) = \begin{bmatrix} p_1^{n-1} & p_2^{n-1} & p_3^{n-1} & p_4^{n-1} \\ p_1^{n-2} & p_2^{n-2} & p_3^{n-2} & p_4^{n-2} \\ p_1^{n-3} & p_2^{n-3} & p_3^{n-3} & p_4^{n-3} \\ p_1^{n-4} & p_2^{n-4} & p_3^{n-4} & p_4^{n-4} \end{bmatrix} ,$$

$$\underline{P}_4^{-1}(0) = (\pi_{ij}) = \begin{bmatrix} p_1^3 & p_2^3 & p_3^3 & p_4^3 \\ p_1^2 & p_2^2 & p_3^2 & p_4^2 \\ p_1 & p_2 & p_3 & p_4 \\ 1 & 1 & 1 & 1 \end{bmatrix}^{-1} , \qquad \text{(III.11)}$$

and p_1, p_2, p_3, p_4 are the roots of

$$p^4 + x_1 p^3 + x_2 p^2 + x_1 p + 1 = 0 \quad . \qquad \text{(III.12)}$$

From the symmetry of the coefficients in (III.12), it is clear that we may write

$$p_1 = \varphi_1,\; p_2 = \frac{1}{\varphi_1},\; p_3 = \varphi_2 \;\text{ and }\; p_4 = \frac{1}{\varphi_2}\;, \tag{III.13}$$

and by [11]

$$\left.\begin{aligned}
\varphi_1 + \frac{1}{\varphi_1} + \varphi_2 + \frac{1}{\varphi_2} &= -x_1 \\
&= \frac{(1+\rho_1^2-2T)}{\rho_1} + \frac{(1+\rho_2^2-2S)}{\rho_2} \\
\text{and} \qquad & \\
(\varphi_1 + \frac{1}{\varphi_1})(\varphi_2 + \frac{1}{\varphi_2}) &= x_2 - 2 \\
&= \frac{(1+\rho_1^2-2T)(1+\rho_2^2-2S)}{\rho_1\rho_2} - \frac{U^2}{\rho_1\rho_2} \;.
\end{aligned}\right\} \tag{III.14}$$

The transformation (III.14) may be considered as consisting of the successive transformations

$$\left.\begin{aligned}
2(P+Q) &= \frac{(1+\rho_1^2-2T)}{\rho_1} + \frac{(1+\rho_2^2-2S)}{\rho_2} \;, \\
4\,P\,Q &= \frac{(1+\rho_1^2-2T)(1+\rho_2^2-2S)}{\rho_1\rho_2} - \frac{1}{\rho_1\rho_2}\left(\frac{T}{r_1} + \frac{S}{r_2}\right)^2 \;,
\end{aligned}\right\} \tag{III.15}$$

(where we have put $U = \dfrac{v(r_1+r_2)-Tr_2-Sr_1}{r_1r_2}$, and $v = 0$ following Appendix I)

and

$$\left.\begin{aligned}
\varphi_1 + \frac{1}{\varphi_1} &= 2P \;, \\
\varphi_2 + \frac{1}{\varphi_2} &= 2Q \;.
\end{aligned}\right\} \tag{III.16}$$

Transformation (III.15) will map paths in the T and S planes, parallel to the imaginary axes, into paths in the P and Q planes which, by the proper choice of branches in transformation (III.16), will be mapped into the interior of the unit circles of the φ_1 and φ_2 planes. Thus we may take $|\varphi_1| < 1$ and $|\varphi_2| < 1$ except in the critical cases where equality

will hold.

Therefore, since each element of the original determinant $|\underline{A}|$ is of bounded modulus as $n \to \infty$ (see (III.8) and (III.9)), $|\varphi_1| < 1$, $|\varphi_2| < 1$, π_1 and π_2 are independent of n (see (III.9)), and m is $O(1)$ with respect to n, we are justified in using the method of Appendix II to approximate $|\underline{A}|$.

A typical element of $\underline{B}^{(0)}_{2,4}\ \underline{P}_4(n-4)\ \underline{P}_4^{-1}(0)$ in (III.10) is

$$e_{st} = \sum_{i=1}^{2} \sum_{j=1}^{4} \beta_{sj}\, p_{2i}^{n-j}\, \pi_{2i,t} + \sum_{i=1}^{2} \sum_{j=1}^{4} \beta_{sj}\, p_{2i-1}^{n-j}\, \pi_{(2i-1),t}\,.$$

Since $p_1 = \varphi_1$, $|\varphi_1| < 1$ and $p_3 = \varphi_2$, $|\varphi_2| < 1$,

$$e_{st} \sim \sum_{i=1}^{2} \sum_{j=1}^{4} \beta_{sj}\, p_{2i}^{n-j}\, \pi_{2i,t} \qquad \text{as} \quad n \to \infty\,,$$

where the terms omitted are relatively $O(\epsilon^n)$ for some $|\epsilon| < 1$. Hence we may write

$$\underline{B}\ \underline{P}_4(n-4)\ \underline{P}_4^{-1}(0)$$

$$\sim \underline{B} \begin{bmatrix} p_2^{n-1} & p_4^{n-1} \\ p_2^{n-2} & p_4^{n-2} \\ p_2^{n-3} & p_4^{n-3} \\ p_2^{n-4} & p_4^{n-4} \end{bmatrix} \begin{bmatrix} \pi_{21} & \pi_{22} & \pi_{23} & \pi_{24} \\ \pi_{41} & \pi_{42} & \pi_{43} & \pi_{44} \end{bmatrix}. \qquad \text{(III.17)}$$

Thus, using (II.15) and substituting (III.17) in (III.10), $|\underline{A}|$ may be approximated by

$$|\underline{A}| \sim \rho_1^n \rho_2^n \left\{ p_2^{n-4} \, p_4^{n-4} (-1) \prod_{1 \leq j < i \leq 4} (p_i - p_j) \right.$$

$$\left. \times \left| \begin{array}{cccc} & \underline{B} & & \\ \hdashline \pi_{11} & \pi_{12} & \pi_{13} & \pi_{14} \\ \pi_{31} & \pi_{32} & \pi_{33} & \pi_{34} \end{array} \right| \left| \begin{array}{cccc} \pi_{21} & \pi_{22} & \pi_{23} & \pi_{24} \\ \pi_{41} & \pi_{42} & \pi_{43} & \pi_{44} \\ \hdashline & \underline{H} & & \end{array} \right| \right\}. \qquad \text{(III.18)}$$

The elements, π_{ij}, of $\underline{P}_4^{-1}(0)$ are of the following form (see Aitken [1])

$(\pi_{11}, \pi_{12}, \pi_{13}, \pi_{14})$

$$= \frac{1}{(p_1-p_2)(p_1-p_3)(p_1-p_4)} [1, -(p_2+p_3+p_4), p_2p_3+p_2p_4+p_3p_4, -p_2p_3p_4] \,,$$

$(\pi_{21}, \pi_{22}, \pi_{23}, \pi_{24})$

$$= \frac{1}{(p_2-p_1)(p_2-p_3)(p_2-p_4)} [1, -(p_1+p_3+p_4), p_1p_3+p_1p_4+p_3p_4, -p_1p_3p_4] \,,$$

and similarly for $(\pi_{31}, \pi_{32}, \pi_{33}, \pi_{34})$, $(\pi_{41}, \pi_{42}, \pi_{43}, \pi_{44})$.

Then, following the reduction procedure of Appendix II, the two determinants in (III.18) become

$$\left| \begin{array}{cccc} & \underline{B} & & \\ \hdashline \pi_{11} & \pi_{12} & \pi_{13} & \pi_{14} \\ \pi_{31} & \pi_{32} & \pi_{33} & \pi_{34} \end{array} \right| = \frac{1}{(p_1-p_2)(p_1-p_3)(p_1-p_4)(p_3-p_2)(p_3-p_4)}$$

$$\times \left| \begin{array}{cccc} & & \underline{B} & \\ \hdashline 1 & -(p_2+p_4) & p_2p_4 & 0 \\ 0 & 1 & -(p_2+p_4) & p_2p_4 \end{array} \right|$$

and

$$\left|\begin{array}{cccc} \pi_{21} & \pi_{22} & \pi_{23} & \pi_{24} \\ \pi_{41} & \pi_{42} & \pi_{43} & \pi_{44} \\ \hline & & \underline{H} & \end{array}\right| = \frac{1}{(p_2-p_1)(p_2-p_3)(p_4-p_1)(p_4-p_2)(p_4-p_3)}$$

$$\times \left|\begin{array}{cccc} 1 & -(p_1+p_3) & p_1p_3 & 0 \\ 0 & 1 & -(p_1+p_3) & p_1p_3 \\ \hline & & \underline{H} & \end{array}\right| .$$

Using the transformation (II.24), we get

$$a_1 = -(\varphi_1+\varphi_2) \quad \text{and} \quad a_2 = \varphi_1 \varphi_2 \; ; \tag{III.19}$$

and, from (III.13), we have

$$p_2p_4 = \frac{1}{a_2}, \quad -(p_2+p_4) = \frac{a_1}{a_2}, \quad p_1p_3 = a_2 \quad \text{and} \quad -(p_1+p_3) = a_1 .$$

Thus (III.18) becomes

$$|\underline{A}| \sim \rho_1^n\rho_2^n \left\{ (-1)a_2^{-(n-4)} \, [(p_1-p_2)(p_2-p_3)(p_3-p_4)(p_1-p_4)]^{-1} \right.$$

$$\left. \times \frac{1}{a_2^2} \left|\begin{array}{cccc} & \underline{B} & & \\ \hline a_2 & a_1 & 1 & 0 \\ 0 & a_2 & a_1 & 1 \end{array}\right| \; \left|\begin{array}{cccc} 1 & a_1 & a_2 & 0 \\ 0 & 1 & a_1 & a_2 \\ \hline & \underline{H} & & \end{array}\right| \right\} . \tag{III.20}$$

Expressing $(p_1-p_2)(p_2-p_3)(p_3-p_4)(p_1-p_4)$ in terms of a_1 and a_2, we have

$$(p_1-p_2)(p_3-p_4) = (\varphi_1 - \frac{1}{\varphi_1})(\varphi_2 - \frac{1}{\varphi_2})$$

$$= \varphi_1\varphi_2 - \frac{1}{\varphi_1\varphi_2}(\varphi_1+\varphi_2)^2 + 2 + \frac{1}{\varphi_1\varphi_2}$$

$$= a_2 - \frac{1}{a_2}(a_1)^2 + 2 + \frac{1}{a_2}$$

$$= \frac{1}{a_2}(1-a_1+a_2)(1+a_1+a_2)$$

and

$$(p_1-p_4)(p_2-p_3) = (\varphi_1 - \frac{1}{\varphi_2})(\frac{1}{\varphi_1} - \varphi_2)$$

$$= 2 - \varphi_1\varphi_2 - \frac{1}{\varphi_1\varphi_2}$$

$$= 2 - a_2 - \frac{1}{a_2}$$

$$= \frac{-1}{a_2}(1-a_2)^2 \quad .$$

Thus

$$|\underline{A}| \sim \frac{\rho_1^n \rho_2^n}{a_2^{n-4}(1-a_1+a_2)(1+a_1+a_2)(1-a_2)^2}$$

$$\times \begin{vmatrix} \beta_{11} & \beta_{12} & 1 & 0 \\ \beta_{21} & x_2 & x_1 & 1 \\ a_2 & a_1 & 1 & 0 \\ 0 & a_2 & a_1 & 1 \end{vmatrix} \begin{vmatrix} 1 & a_1 & a_2 & 0 \\ 0 & 1 & a_1 & a_2 \\ 1 & x_1 & x_2 & \beta_{21} \\ 0 & 1 & \beta_{12} & \beta_{11} \end{vmatrix} , \qquad \text{(III.21)}$$

since $\beta_{13} = \beta_{24} = 1$, $\beta_{14} = 0$, $\beta_{22} = x_2$ and $\beta_{23} = x_1$.

From (III.14) and (III.19)

$$\left.\begin{aligned} x_1 &= -[\varphi_1 + \varphi_2 + \frac{\varphi_1+\varphi_2}{\varphi_1\,\varphi_2}] \\ &= \frac{a_1}{a_2}(1 + a_2) \\ \text{and} \\ x_2 &= \varphi_1\varphi_2 + \frac{\varphi_1^2 + \varphi_2^2}{\varphi_1\,\varphi_2} + \frac{1}{\varphi_1\varphi_2} + 2 \\ &= \frac{1}{a_2}(1 + a_1^2 + a_2^2) \quad . \end{aligned}\right\} \qquad \text{(III.22)}$$

Finally, using (III.22) in (III.21), we have

$$|\underline{A}| \sim \frac{\rho_1^n \; \rho_2^n \; [a_2(\beta_{11}-a_2) - a_2^2(\beta_{12}-a_1)(\beta_{21}-a_1)]^2}{a_2^n \quad (1-a_1+a_2)(1+a_1+a_2)(1-a_2)^2} \quad , \qquad (III.23)$$

where terms which become exponentially small have been omitted.

B.) Fitted Means

In order to obtain the joint moment-generating function of C^*, D^* and E^* (3.7) the determinant of $\underline{B}$ must be evaluated. The approximate value of $|\underline{B}|$ is the key to the problem of finding the distribution of the product-moment correlation for the fitted means case.

$\underline{B}$ is the nonsingular, $2n \times 2n$, partitioned matrix,

$$\underline{B} = \left[\begin{array}{c|c} \underline{R}_1 & \underline{R}_3 \\ \hline \underline{R}_3 & \underline{R}_2 \end{array}\right] , \qquad (III.24)$$

with $n \times n$ submatrices

$$\underline{R}_1 = \begin{bmatrix} a+\alpha & -\rho_1+\alpha & \alpha & \cdots & \alpha & \alpha \\ -\rho_1+\alpha & b+\alpha & -\rho_1+\alpha & \cdots & \alpha & \alpha \\ \alpha & -\rho_1+\alpha & b+\alpha & \cdots & \alpha & \alpha \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \alpha & \alpha & \alpha & \cdots & b+\alpha & -\rho_1+\alpha \\ \alpha & \alpha & \alpha & \cdots & -\rho_1+\alpha & a+\alpha \end{bmatrix} , \qquad (III.25)$$

$$\underline{R}_2 = \begin{bmatrix} c+\beta & -\rho_2+\beta & \beta & \cdots & \beta & \beta \\ -\rho_2+\beta & d+\beta & -\rho_2+\beta & \cdots & \beta & \beta \\ \beta & -\rho_2+\beta & d+\beta & \cdots & \beta & \beta \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \beta & \beta & \beta & \cdots & d+\beta & -\rho_2+\beta \\ \beta & \beta & \beta & \cdots & -\rho_2+\beta & c+\beta \end{bmatrix} \qquad (III.26)$$

and

$$\underline{R}_3 = \begin{bmatrix} -U+\gamma & \gamma & \gamma & \cdots & \gamma & \gamma \\ \gamma & -U+\gamma & \gamma & \cdots & \gamma & \gamma \\ \gamma & \gamma & -U+\gamma & \cdots & \gamma & \gamma \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \gamma & \gamma & \gamma & \cdots & -U+\gamma & \gamma \\ \gamma & \gamma & \gamma & \cdots & \gamma & -U+\gamma \end{bmatrix} , \qquad \text{(III.27)}$$

where

$$\left.\begin{aligned} a &= 1-2T , \\ b &= 1+\rho_1^2-2T , \\ c &= 1-2S , \\ d &= 1+\rho_2^2-2S , \\ &\text{and} \\ \alpha &= \frac{2T}{n} , \\ \beta &= \frac{2S}{n} , \\ \gamma &= \frac{U}{n} . \end{aligned}\right\} \qquad \text{(III.28)}$$

In order to be able to use the determinant-evaluation method of Appendix II we must transform $|\underline{B}|$ to the form (II.1). To do this we shall use the bordering technique of Aitken [1] and Muir [10] followed by a series of pre- and post-multiplications.

The determinant $|\underline{B}|$ may be written as the $(2n+2)\times(2n+2)$ bordered determinant,

$$
|\underline{C}_1| = \left|\begin{array}{c|ccccc|ccccc|c}
1 & \alpha & \alpha & \ldots & \alpha & & \gamma & \gamma & \ldots & \gamma & & 0 \\
\hline
0 & & & & & & & & & & & 0 \\
\vdots & & & \underline{R}_1 & & & & & \underline{R}_3 & & & \vdots \\
0 & & & & & & & & & & & 0 \\
\hline
0 & & & & & & & & & & & 0 \\
\vdots & & & \underline{R}_3 & & & & & \underline{R}_2 & & & \vdots \\
0 & & & & & & & & & & & 0 \\
\hline
0 & \gamma & \gamma & \ldots & \gamma & & \alpha & \alpha & \ldots & \alpha & & 1
\end{array}\right| . \qquad \text{(III.29)}
$$

Premultiplication of $\underline{C}_1$ by the $(2n+2)\times(2n+2)$ partitioned matrix

$$
\underline{I}_1 = \left[\begin{array}{cccccc|cccccc}
1 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\
-1 & 1 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\
-1 & 0 & 1 & \ldots & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
-1 & 0 & 0 & \ldots & 0 & 1 & 0 & 0 & \ldots & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & \ldots & 0 & 0 & 1 & 0 & \ldots & 0 & 0 & -1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & 1 & 0 & -1 \\
0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & 0 & 1 & -1 \\
0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 1
\end{array}\right] ,
$$

yields,

$|\underline{C}_2| = |\underline{I}_1\underline{C}_1| =$

$$
\left|\begin{array}{c|ccccccc|ccccccc|c}
1 & \alpha & \alpha & \alpha & \cdots & \alpha & \alpha & \alpha & \gamma & \gamma & \gamma & \cdots & \gamma & \gamma & \gamma & 0 \\
\hline
-1 & a & -\rho_1 & 0 & \cdots & 0 & 0 & 0 & -U & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
-1 & -\rho_1 & b & -\rho_1 & \cdots & 0 & 0 & 0 & 0 & -U & 0 & \cdots & 0 & 0 & 0 & 0 \\
-1 & 0 & -\rho_1 & b & \cdots & 0 & 0 & 0 & 0 & 0 & -U & \cdots & 0 & 0 & 0 & 0 \\
\vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots \\
-1 & 0 & 0 & 0 & \cdots & b & -\rho_1 & 0 & 0 & 0 & 0 & \cdots & -U & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & \cdots & -\rho_1 & b & -\rho_1 & 0 & 0 & 0 & \cdots & 0 & -U & 0 & 0 \\
-1 & 0 & 0 & 0 & \cdots & 0 & -\rho_1 & a & 0 & 0 & 0 & \cdots & 0 & 0 & -U & 0 \\
\hline
0 & -U & 0 & 0 & \cdots & 0 & 0 & 0 & c & -\rho_2 & 0 & \cdots & 0 & 0 & 0 & -1 \\
0 & 0 & -U & 0 & \cdots & 0 & 0 & 0 & -\rho_2 & d & -\rho_2 & \cdots & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & -U & \cdots & 0 & 0 & 0 & 0 & -\rho_2 & d & \cdots & 0 & 0 & 0 & -1 \\
\vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & -U & 0 & 0 & 0 & 0 & 0 & \cdots & d & -\rho_2 & 0 & -1 \\
0 & 0 & 0 & 0 & \cdots & 0 & -U & 0 & 0 & 0 & 0 & \cdots & -\rho_2 & d & -\rho_2 & -1 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & -U & 0 & 0 & 0 & \cdots & 0 & -\rho_2 & c & -1 \\
\hline
0 & \gamma & \gamma & \gamma & \cdots & \gamma & \gamma & \gamma & \beta & \beta & \beta & \cdots & \beta & \beta & \beta & 1
\end{array}\right|
\tag{III.30}
$$

which we may write as

$$
|\underline{C}_2| = \left|\begin{array}{c|ccc|ccc|c}
1 & \alpha & \cdots & \alpha & \gamma & \cdots & \gamma & 0 \\
\hline
-1 & & & & & & & 0 \\
\vdots & & \underline{Q}_1 & & & \underline{Q}_3 & & \vdots \\
-1 & & & & & & & 0 \\
\hline
0 & & & & & & & -1 \\
\vdots & & \underline{Q}_3 & & & \underline{Q}_2 & & \vdots \\
0 & & & & & & & -1 \\
\hline
0 & \gamma & \cdots & \gamma & \beta & \cdots & \beta & 1
\end{array}\right| ,
\tag{III.31}
$$

where $\underline{Q}_1$ and $\underline{Q}_2$ are defined by (III.2) and (III.3), respectively,

and $\underline{Q}_3$ is a scaler $(n \times n)$ matrix with diagonal elements $(-U)$.

Premultiplying $\underline{C}_2$ by the $(2n+2)\times(2n+2)$ matrix

$$\underline{I}_2 = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & 1 & 0 \\ -\frac{\gamma}{\alpha} & 0 & \cdots & 0 & 1 \end{bmatrix},$$

we get

$$|\underline{C}_3| = |\underline{I}_2\underline{C}_2| = \left|\begin{array}{c|ccc|ccc|c} 1 & \alpha & \cdots & \alpha & \gamma & \cdots & \gamma & 0 \\ \hline -1 & & & & & & & 0 \\ \vdots & & \underline{Q}_1 & & & \underline{Q}_3 & & \vdots \\ -1 & & & & & & & 0 \\ \hline 0 & & & & & & & -1 \\ \vdots & & \underline{Q}_3 & & & \underline{Q}_2 & & \vdots \\ 0 & & & & & & & -1 \\ \hline -\frac{\gamma}{\alpha} & 0 & \cdots & 0 & \beta-\frac{\gamma^2}{\alpha} & \cdots & \beta-\frac{\gamma^2}{\alpha} & 1 \end{array}\right|, \qquad \text{(III.32)}$$

and premultiplying $\underline{C}_3$ by the $(2n+2)\times(2n+2)$ matrix

$$\underline{I}_3 = \begin{bmatrix} 1 & 0 & \cdots & 0 & -\gamma[\beta-\frac{\gamma^2}{\alpha}]^{-1} \\ 0 & 1 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix},$$

we have

$$|\underline{C}_4| = |\underline{I}_3\underline{C}_3| = \left|\begin{array}{c|ccc|ccc|c}
x_3 & \alpha & \cdots & \alpha & 0 & \cdots & 0 & x_4 \\
\hline
-1 & & & & & & & 0 \\
\vdots & & \underline{Q}_1 & & & \underline{Q}_3 & & \vdots \\
-1 & & & & & & & 0 \\
\hline
0 & & & & & & & -1 \\
\vdots & & \underline{Q}_3 & & & \underline{Q}_2 & & \vdots \\
0 & & & & & & & -1 \\
\hline
x_1 & 0 & \cdots & 0 & x_2 & \cdots & x_2 & 1
\end{array}\right| , \tag{III.33}$$

where

$$x_1 = -\frac{\gamma}{\alpha} , \qquad x_2 = \beta - \frac{\gamma^2}{\alpha} ,$$

$$x_3 = \frac{\alpha\beta}{\alpha\beta-\gamma^2} \quad \text{and} \quad x_4 = \frac{-\gamma}{\beta-\frac{\gamma^2}{\alpha}} .$$

To procure more zero elements in the first and last rows of $|\underline{C}_4|$, we proceed as follows:

firstly, solving for λ_1 and λ_2 in the two equations

$$\alpha - \lambda_1(b-2\rho_1) - \lambda_2(-U) = 0$$

and

$$-\lambda_1(-U) - \lambda_2(d-2\rho_2) = 0 ,$$

we get

$$\lambda_1 = \frac{\alpha(d-2\rho_2)}{[(b-2\rho_1)(d-2\rho_2) - U^2]}$$

and

$$\lambda_2 = \frac{\alpha U}{[(b-2\rho_1)(d-2\rho_2) - U^2]} .$$

Then premultiplication of $\underline{C}_4$, (III.33) by the $(2n+2)\times(2n+2)$ matrix

$$\underline{I}_4 = \begin{bmatrix} 1 & -\lambda_1 & -\lambda_1 & \cdots & -\lambda_1 & -\lambda_2 & \cdots & -\lambda_2 & -\lambda_2 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 1 \end{bmatrix},$$

gives

$$|\underline{C}_5| = |\underline{I}_4\underline{C}_4| = [(b-2\rho_1)(d-2\rho_2) - U^2]^{-1}$$

$$\times \left|\begin{array}{c|ccccc|ccccc|c} x_5 & x_6 & 0 & \cdots & 0 & x_6 & x_7 & 0 & \cdots & 0 & x_7 & x_8 \\ \hline -1 & & & & & & & & & & & 0 \\ \vdots & & & \underline{Q}_1 & & & & & \underline{Q}_3 & & & \vdots \\ -1 & & & & & & & & & & & 0 \\ \hline 0 & & & & & & & & & & & -1 \\ \vdots & & & \underline{Q}_3 & & & & & \underline{Q}_2 & & & \vdots \\ 0 & & & & & & & & & & & -1 \\ \hline x_1 & 0 & 0 & \cdots & 0 & 0 & x_2 & x_2 & \cdots & x_2 & x_2 & 1 \end{array}\right|, \qquad \text{(III.34)}$$

where $[(b-2\rho_1)(d-2\rho_2) - U^2]^{-1}$ is factored from the first row and where

$$x_5 = [(b-2\rho_1)(d-2\rho_2) - U^2][x_3 + n\lambda_1]$$

$$= x_3[(b-2\rho_1)(d-2\rho_2) - U^2] + n\alpha(d-2\rho_2) \quad ,$$

$$x_6 = [(b-2\rho_1)(d-2\rho_2)-U^2][\alpha-\lambda_1(a-\rho_1) + \lambda_2 U]$$

$$= \alpha[(b-2\rho_1)(d-2\rho_2)-U^2] - \alpha(d-2\rho_2)(a-\rho_1) + \alpha U^2$$

$$= \alpha(d-2\rho_2)[b-2\rho_1 - a + \rho_1]$$

$$= \alpha\rho_1(\rho_1-1)(d-2\rho_2) \quad ,$$

$$x_7 = [(b-2\rho_1)(d-2\rho_2)-U^2][\lambda_1 U - \lambda_2(c-\rho_2)]$$

$$= \alpha(d-2\rho_2)U - \alpha U(c-\rho_2)$$

$$= \alpha\ U[d-2\rho_2-c+\rho_2]$$

$$= \alpha\rho_2(\rho_2-1)U$$

and

$$x_8 = [(b-2\rho_1)(d-2\rho_2)-U^2][x_4 + n\lambda_2]$$

$$= x_4[(b-2\rho_1)(d-2\rho_2)-U^2] + n\ \alpha\ U \quad .$$

Secondly, from the two equations

$$x_2 - \lambda_3(d-2\rho_2) - \lambda_4(-U) = 0$$

and

$$- \lambda_3(-U) - \lambda_4(b-2\rho_1) = 0$$

we have that

$$\lambda_3 = \frac{x_2(b-2\rho_1)}{[(b-2\rho_1)(d-2\rho_2)-U^2]}$$

and

$$\lambda_4 = \frac{x_2 U}{[(b-2\rho_1)(d-2\rho_2)-U^2]} \quad .$$

Hence, premultiplication of $\underline{C}_5$ (III.34) by the $(2n+2)\times(2n+2)$ matrix

$$\underline{I}_5 = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & -\lambda_4 & -\lambda_4 & \cdots & -\lambda_4 & -\lambda_3 & \cdots & -\lambda_3 & -\lambda_3 & 1 \end{bmatrix},$$

gives $\underline{C}_6$,

$$|\underline{C}_6| = |\underline{I}_5\underline{C}_5| = [(b-2\rho_1)(d-2\rho_2)-U^2]^{-2}$$

$$\times \left| \begin{array}{c|ccccc|ccccc|c} x_5 & x_6 & 0 & \cdots & 0 & x_6 & x_7 & 0 & \cdots & 0 & x_7 & x_8 \\ \hline -1 & & & & & & & & & & & 0 \\ \vdots & & & \underline{Q}_1 & & & & & \underline{Q}_3 & & & \vdots \\ -1 & & & & & & & & & & & 0 \\ \hline 0 & & & & & & & & & & & -1 \\ \vdots & & & \underline{Q}_3 & & & & & \underline{Q}_2 & & & \vdots \\ 0 & & & & & & & & & & & -1 \\ \hline x_9 & x_{10} & 0 & \cdots & 0 & x_{10} & x_{11} & 0 & \cdots & 0 & x_{11} & x_{12} \end{array} \right|, \qquad \text{(III.35)}$$

where $[(b-2\rho_1)(d-2\rho_2)-U^2]^{-1}$ is factored from the last row, and where

$$x_9 = [(b-2\rho_1)(d-2\rho_2)-U^2][n\lambda_4 + x_1]$$

$$= nx_2U + x_1[(b-2\rho_1)(d-2\rho_2)-U^2] \quad ,$$

$$x_{10} = [(b-2\rho_1)(d-2\rho_2)-U^2][-\lambda_4(a-\rho_1)+\lambda_3 U]$$

$$= -x_2 U(a-\rho_1) + x_2(b-2\rho_1)U$$

$$= x_2 U[b-2\rho_1-a + \rho_1]$$

$$= x_2\rho_1(\rho_1-1)U \quad ,$$

$$x_{11} = [(b-2\rho_1)(d-2\rho_2)-U^2][\lambda_4 U-\lambda_3(c-\rho_2)+x_2]$$

$$= x_2U^2-x_2(b-2\rho_1)(c-\rho_2)+x_2[(b-2\rho_1)(d-2\rho_2)-U^2]$$

$$= x_2(b-2\rho_1)[d-2\rho_2-c+\rho_2]$$

$$= x_2\rho_2(\rho_2-1)(b-2\rho_1)$$

and

$$x_{12} = [(b-2\rho_1)(d-2\rho_2)-U^2][n\lambda_3+1]$$

$$= nx_2(b-2\rho_1)+[(b-2\rho_1)(d-2\rho_2)-U^2] \quad .$$

To replace most of the (-1)'s by zeros in the first and last columns of $|\underline{C}_6|$, we shall employ the same technique as in the derivation of $|\underline{C}_6|$, (III.35), from $|\underline{C}_4|$, (III.33). Solving the equations

$$-1 - \mu_1(b-2\rho_1) - \mu_2(-U) = 0$$

and

$$- \mu_1(-U) - \mu_2(d-2\rho_2) = 0 \quad ,$$

we get

$$\mu_1 = \frac{-(d-2\rho_2)}{[(b-2\rho_1)(d-2\rho_2)-U^2]}$$

and

$$\mu_2 = \frac{-\,U}{[(b-2\rho_1)(d-2\rho_2)-U^2]} \quad .$$

Post-multiplying $\underline{C}_6$ (III.35) by the $(2n+2)\times(2n+2)$ matrix

$$\underline{I}_6 = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -\mu_1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ -\mu_1 & 0 & 1 & \cdots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ -\mu_1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -\mu_2 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ -\mu_2 & 0 & 0 & \cdots & 1 & 0 & 0 \\ -\mu_2 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{bmatrix} ,$$

the determinant of the product becomes

$$|\underline{C}_7| = |\underline{C}_6\underline{I}_6| = [(b-2\rho_1)(d-2\rho_2)\ -U^2]^{-3}$$

$$\times \left| \begin{array}{c|ccccc|ccccc|c} y_1 & x_6 & 0 & \cdots & 0 & x_6 & x_7 & 0 & \cdots & 0 & x_7 & x_8 \\ \hline y_2 & & & & & & & & & & & 0 \\ 0 & & & & & & & & & & & 0 \\ \vdots & & & \underline{Q}_1 & & & & & \underline{Q}_3 & & & \vdots \\ 0 & & & & & & & & & & & 0 \\ y_2 & & & & & & & & & & & 0 \\ \hline y_3 & & & & & & & & & & & -1 \\ 0 & & & & & & & & & & & -1 \\ \vdots & & & \underline{Q}_3 & & & & & \underline{Q}_2 & & & \vdots \\ 0 & & & & & & & & & & & -1 \\ y_3 & & & & & & & & & & & -1 \\ \hline y_4 & x_{10} & 0 & \cdots & 0 & x_{10} & x_{11} & 0 & \cdots & 0 & x_{11} & x_{12} \end{array} \right| , \qquad \text{(III.36)}$$

where $[(b-2\rho_1)(d-2\rho_2)-U^2]^{-1}$ is factored from the first column and so

$$y_1 = [(b-2\rho_1)(d-2\rho_2)-U^2][x_5-\mu_1(2x_6)-\mu_2(2x_7)]$$

$$= x_5[(b-2\rho_1)(d-2\rho_2)-U^2] + 2[(d-2\rho_2)x_6 + Ux_7] ,$$

$$y_2 = [(b-2\rho_1)(d-2\rho_2)-U^2][-1-\mu_1(a-\rho_1) + \mu_2 U]$$

$$= -[(b-2\rho_1)(d-2\rho_2)-U^2] + (d-2\rho_2)(a-\rho_1)-U^2$$

$$= -(d-2\rho_2)[b-2\rho_1-a+\rho_1]$$

$$= -\rho_1(\rho_1-1)(d-2\rho_2) ,$$

$$y_3 = [(b-2\rho_1)(d-2\rho_2)-U^2][\mu_1 U-\mu_2(c-\rho_2)]$$

$$= -(d-2\rho_2)U + (c-\rho_2)U$$

$$= -\rho_2(\rho_2-1)U$$

and

$$y_4 = [(b-2\rho_1)(d-2\rho_2)-U^2][x_9-\mu_1(2x_{10}) - \mu_2(2x_{11})]$$

$$= x_9[(b-2\rho_1)(d-2\rho_2)-U^2] + 2[(d-2\rho_2)x_{10} + Ux_{11}] .$$

Similarly for the last column, solving the equations

$$-\mu_3(-U) - \mu_4(b-2\rho_1) = 0$$

$$-1 -\mu_3(d-2\rho_2) - \mu_4(-U) = 0 ,$$

we obtain

$$\mu_3 = \frac{-(b-2\rho_1)}{[(b-2\rho_1)(d-2\rho_2)-U^2]}$$

and

$$\mu_4 = \frac{-U}{[(b-2\rho_1)(d-2\rho_2)-U^2]} \quad .$$

Hence, postmultiplying $\underline{C}_7$ (III.33) by the $(2n+2)\times(2n+2)$ matrix

$$\underline{I}_7 = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & -\mu_4 \\ 0 & 0 & 1 & \cdots & 0 & 0 & -\mu_4 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 0 & 0 & -\mu_4 \\ 0 & 0 & 0 & \cdots & 0 & 0 & -\mu_3 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 1 & 0 & -\mu_3 \\ 0 & 0 & 0 & \cdots & 0 & 1 & -\mu_3 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{bmatrix} ,$$

we have, as the determinant of the product,

$$|\underline{C}_8| = |\underline{C}_7\underline{I}_7| = [(b-2\rho_1)(d-2\rho_2)-U^2]^{-4}$$

$$\times \left| \begin{array}{c|ccccc|ccccc|c} y_1 & x_6 & 0 & \cdots & 0 & x_6 & x_7 & 0 & \cdots & 0 & x_7 & y_5 \\ \hline y_2 & & & & & & & & & & & y_6 \\ 0 & & & & & & & & & & & 0 \\ \vdots & & & \underline{\Omega}_1 & & & & & \underline{\Omega}_3 & & & \vdots \\ 0 & & & & & & & & & & & 0 \\ y_2 & & & & & & & & & & & y_6 \\ \hline y_3 & & & & & & & & & & & y_7 \\ 0 & & & & & & & & & & & 0 \\ \vdots & & & \underline{\Omega}_3 & & & & & \underline{\Omega}_2 & & & \vdots \\ 0 & & & & & & & & & & & 0 \\ y_3 & & & & & & & & & & & y_7 \\ \hline y_4 & x_{10} & 0 & \cdots & 0 & x_{10} & x_{11} & 0 & \cdots & 0 & x_{11} & y_8 \end{array} \right| , \qquad \text{(III.37)}$$

where $[(b-2\rho_1)(d-2\rho_2)-U^2]^{-1}$ is factored from the last column and where

$$y_5 = [(b-2\rho_1)(d-2\rho_2)-U^2][-\mu_4(2x_6)-\mu_3(2x_7) + x_8]$$

$$= 2[Ux_6+(b-2\rho_1)x_7] + x_8[(b-2\rho_1)(d-2\rho_2)-U^2] \quad ,$$

$$y_6 = [(b-2\rho_1)(d-2\rho_2)-U^2][-\mu_4(a-\rho_1) + \mu_3 U]$$

$$= U(a-\rho_1)-(b-2\rho_1)U$$

$$= -\rho_1(\rho_1-1)U \quad ,$$

$$y_7 = [(b-2\rho_1)(d-2\rho_2)-U^2][\mu_4 U-\mu_3(c-\rho_2)-1]$$

$$= -U^2 + (b-2\rho_1)(c-\rho_2)-[(b-2\rho_1)(d-2\rho_2)-U^2]$$

$$= -(b-2\rho_1)[d-2\rho_2-c+\rho_2]$$

$$= -\rho_2(\rho_2-1)(b-2\rho_1)$$

and

$$y_8 = [(b-2\rho_1)(d-2\rho_2-U^2][-\mu_4(2x_{10})-\mu_3(2x_{11}) + x_{12}]$$

$$= 2[Ux_{10}+(b-2\rho_1)x_{11}] + x_{12}[(b-2\rho_1)(d-2\rho_2)-U^2] \quad .$$

$$|\underline{B}| = |\underline{C}_8|$$

$$= [(b-2\rho_1)(d-2\rho_2)-U^2]^{-4}$$

$$\times \left| \begin{array}{c|ccccc|ccccc|c}
y_1 & x_6 & 0 & \cdots & 0 & x_6 & x_7 & 0 & \cdots & 0 & x_7 & y_5 \\ \hline
y_2 & & & & & & & & & & & y_6 \\
0 & & & & & & & & & & & 0 \\
\vdots & & & \underline{Q}_1 & & & & & \underline{Q}_3 & & & \vdots \\
0 & & & & & & & & & & & 0 \\
y_2 & & & & & & & & & & & y_6 \\ \hline
y_3 & & & & & & & & & & & y_7 \\
0 & & & & & & & & & & & 0 \\
\vdots & & & \underline{Q}_3 & & & & & \underline{Q}_2 & & & \vdots \\
0 & & & & & & & & & & & 0 \\
y_3 & & & & & & & & & & & y_7 \\ \hline
y_4 & x_{10} & 0 & \cdots & 0 & x_{10} & x_{11} & 0 & \cdots & 0 & x_{11} & y_8
\end{array} \right| , \qquad \text{(III.38)}$$

where, using relations (III.28),

$$x_6 = \alpha\rho_1(\rho_1-1)(d-2\rho_2) = 0(\tfrac{1}{n}) \quad ,$$

$$x_7 = \alpha\rho_2(\rho_2-1)U \qquad = 0(\tfrac{1}{n}) \quad ,$$

$$x_{10} = x_2\rho_1(\rho_1-1)U = (\beta - \frac{\gamma^2}{\alpha})\rho_1(\rho_1-1)U$$

$$= 0(\tfrac{1}{n}) \quad ,$$

$$x_{11} = x_2\rho_2(\rho_2-1)(b-2\rho_1) = (\beta - \frac{\gamma^2}{\alpha})\rho_2(\rho_2-1)(b-2\rho_1)$$

$$= 0(\tfrac{1}{n}) \quad ,$$

$$y_1 = x_5[(b-2\rho_1)(d-2\rho_2)-U^2] + 2[(d-2\rho_2)x_6+Ux_7]$$

$$= x_3[(b-2\rho_1)(d-2\rho_2)-U^2]^2 + n\alpha(d-2\rho_2)[(b-2\rho_1)(d-2\rho_2)-U^2]$$

$$+ 0(\tfrac{1}{n}) \qquad \text{(III.39)}$$

$$= \left[\frac{4TS}{4TS-U^2}\right] [(b-2\rho_1)(d-2\rho_2)-U^2]^2$$

$$+ 2T(d-2\rho_2)[(b-2\rho_1)(d-2\rho_2)-U^2] + 0(\tfrac{1}{n}) \quad ,$$

$$y_2 = -\rho_1(\rho_1-1)(d-2\rho_2) \; ,$$

$$y_3 = -\rho_2(\rho_2-1)U \quad ,$$

$$y_4 = x_9[(b-2\rho_1)(d-2\rho_2)-U^2] + 2[(d-2\rho_2)x_{10} + Ux_{11}]$$

$$= nx_2U[(b-2\rho_1)(d-2\rho_2)-U^2] + x_1[(b-2\rho_1)(d-2\rho_2)-U^2]^2$$

$$+ 0(\tfrac{1}{n})$$

$$= \left[\frac{4TS-U^2}{2T}\right] U[(b-2\rho_1)(d-2\rho_2)-U^2] - \frac{U}{2T} [(b-2\rho_1)(d-2\rho_2)-U^2]^2$$

$$+ 0(\tfrac{1}{n}) \quad , \qquad \text{(III.40)}$$

$$y_5 = 2[Ux_6+(b-2\rho_1)x_7] + x_8[(b-2\rho_1)(d-2\rho_2)-U^2]$$

$$= 0(\tfrac{1}{n}) + x_4[(b-2\rho_1)(d-2\rho_2)-U^2]^2$$

$$+ n\alpha U[(b-2\rho_1)(d-2\rho_2)-U^2]$$

$$= \left[\frac{-2TU}{4TS-U^2}\right] [(b-2\rho_1)(d-2\rho_2)-U^2]^2$$

$$+ 2TU[(b-2\rho_1)(d-2\rho_2)-U^2] + 0(\tfrac{1}{n}) \quad , \qquad \text{(III.41)}$$

$$y_6 = -\rho_1(\rho_1-1)U \; ,$$

$$y_7 = -\rho_2(\rho_2-1)(b-2\rho_1)$$

and

$$y_8 = 2[Ux_{10}+(b-2\rho_1)x_{11}] + x_{12}[(b-2\rho_1)(d-2\rho_2)-U^2]$$

$$= O(\tfrac{1}{n}) + nx_2(b-2\rho_1)[(b-2\rho_1)(d-2\rho_2)-U^2]$$

$$+ [(b-2\rho_1)(d-2\rho_2)-U^2]^2$$

$$= \left[\frac{4TS-U^2}{2T}\right](b-2\rho_1)[(b-2\rho_1)(d-2\rho_2)-U^2]$$

$$+ [(b-2\rho_1)(d-2\rho_2)-U^2]^2 + O(\tfrac{1}{n}) \quad . \tag{III.42}$$

We now expand the $(2n+2)$ determinant of (III.38), which we denote by

$$B_1 = [(b-2\rho_1)(d-2\rho_2)-U^2]^4 \; |\underline{B}| \quad . \tag{III.43}$$

Ignoring terms which are $O(\frac{1}{n})$,

$$B_1 \sim y_1 \, B_2 + (-1)^{2n+3} \, y_5 \, B_3$$

that is

$$B_1 \sim y_1 \, B_2 - y_5 \, B_3 \quad , \tag{III.44}$$

where

$$B_2 = \left| \begin{array}{ccc|ccc|c} & & & & & & y_6 \\ & & & & & & 0 \\ & \underline{Q}_1 & & & \underline{Q}_3 & & \vdots \\ & & & & & & 0 \\ & & & & & & y_6 \\ \hline & & & & & & y_7 \\ & & & & & & 0 \\ & \underline{Q}_3 & & & \underline{Q}_2 & & \vdots \\ & & & & & & 0 \\ & & & & & & y_7 \\ \hline x_{10} \; 0 & \cdots & 0 \; x_{10} & x_{11} \; 0 & \cdots & 0 \; x_{11} & y_8 \end{array} \right|$$

and

$$B_3 = \left| \begin{array}{c|c|c} y_2 & & \\ 0 & & \\ \vdots & \underline{Q}_1 & \underline{Q}_3 \\ 0 & & \\ y_2 & & \\ \hline y_3 & & \\ 0 & & \\ \vdots & \underline{Q}_3 & \underline{Q}_2 \\ 0 & & \\ y_3 & & \\ \hline y_4 & x_{10} \; 0 \; \ldots \; 0 \; x_{10} & x_{11} \; 0 \; \ldots \; 0 \; x_{11} \end{array} \right| .$$

Expanding B_2 and B_3 and ignoring terms which are $0(\frac{1}{n})$, we obtain

$$B_2 \sim y_8 \, |\underline{A}| \tag{III.45}$$

and

$$B_3 \sim y_4 \, |\underline{A}| \;, \tag{III.46}$$

where $|\underline{A}|$,

$$|\underline{A}| = \left| \begin{array}{c|c} \underline{Q}_1 & \underline{Q}_3 \\ \hline \underline{Q}_3 & \underline{Q}_2 \end{array} \right| \;,$$

is the same determinant as encountered in the "Known Means" case (III.1). Thus we have from (III.43), (III.44), (III.45) and (III.46) that

$$|\underline{B}| \sim [(b-2\rho_1)(d-2\rho_2)-U^2]^{-4}$$

$$\times [y_1 y_8 - y_5 y_4] \; |\underline{A}| \quad . \tag{III.47}$$

Using the relations (III.39), (III.40), (III.41) and (III.42), we may write

$$
\begin{aligned}
y_1y_8-y_5y_4 = &\left\{\left[\frac{4TS}{4TS-U^2}\right] [(b-2\rho_1)(d-2\rho_2)-U^2]^2\right.\\
&\left. + 2T(d-2\rho_2)[(b-2\rho_1)(d-2\rho_2)-U^2]\right\}\\
&\times\left\{\left[\frac{4TS-U}{2T}\right] (b-2\rho_1)[(b-2\rho_1)(d-2\rho_2)-U^2]\right.\\
&\left. + [(b-2\rho_1)(d-2\rho_2)-U^2]^2\right\}\\
&- \left\{\left[\frac{-2TU}{4TS-U^2}\right] [(b-2\rho_1)(d-2\rho_2)-U^2]^2\right.\\
&\left. + 2TU[(b-2\rho_1)(d-2\rho_2)-U^2]\right\}\\
&\times\left\{\left[\frac{4TS-U^2}{2T}\right] U[(b-2\rho_1)(d-2\rho_2)-U^2]\right.\\
&\left. - \frac{U}{2T} [(b-2\rho_1)(d-2\rho_2)-U^2]^2\right\} + 0(\tfrac{1}{n})\\
= &[(b-2\rho_1)(d-2\rho_2)-U^2]^2\\
&\times\left\{ 2S(b-2\rho_1)[(b-2\rho_1)(d-2\rho_2)-U^2]\right.\\
&+ \left[\frac{4TS}{4TS-U^2}\right] [(b-2\rho_1)(d-2\rho_2)-U^2]^2\\
&+ [4TS-U^2] (d-2\rho_2)(b-2\rho_1)\\
&+ 2T(d-2\rho_2)[(b-2\rho_1)(d-2\rho_2)-U^2]\\
&+ U^2[(b-2\rho_1)(d-2\rho_2)-U^2]\\
&- \left[\frac{U^2}{4TS-U^2}\right] [(b-2\rho_1)(d-2\rho_2)-U^2]^2\\
&- [4TS-U^2]U^2\\
&\left.+ U^2[(b-2\rho_1)(d-2\rho_2)-U^2]\right\} + 0(\tfrac{1}{n})
\end{aligned}
$$

continued

$$= [(b-2\rho_1)(d-2\rho_2)-U^2]^3$$

$$\times\left\{\left[\frac{4TS}{4TS-U^2}\right][(b-2\rho_1)(d-2\rho_2)-U^2] + 2S(b-2\rho_1)\right.$$

$$+ 2T(d-2\rho_2) + 2U^2 - \left[\frac{U^2}{4TS-U^2}\right][(b-2\rho_1)(d-2\rho_2)-U^2]$$

$$\left. + (4TS-U^2)\right\} + o(\tfrac{1}{n})$$

$$= [(b-2\rho_1)(d-2\rho_2)-U^2]^3$$

$$\times\left\{(b-2\rho_1)[d-2\rho_2+2S] + 2T[d-2\rho_2+2S]\right\} + o(\tfrac{1}{n})$$

$$= [(b-2\rho_1)(d-2\rho_2)-U^2]^3$$

$$\times[d-2\rho_2+2S][b-2\rho_1+2T] + o(\tfrac{1}{n})$$

and using the relations (III.28)

$$y_1y_8-y_5y_4 = [(b-2\rho_1)(d-2\rho_2)-U^2]^3$$

$$\times [1+\rho_2^2-2S-2\rho_2+2S][1+\rho_1^2-2T-2\rho_1+2T] + o(\tfrac{1}{n})$$

$$= [(b-2\rho_1)(d-2\rho_2)-U^2]^3(1-\rho_2)^2(1-\rho_1)^2 + o(\tfrac{1}{n}) \quad .$$

Thus (III.47) becomes

$$|\underline{B}| \sim [(b-2\rho_1)(d-2\rho_2)-U^2]^{-1} (1-\rho_1)^2(1-\rho_2^2)|\underline{A}| \quad . \tag{III.48}$$

From (III.9) and (III.22) we have the relations

$$\left.\begin{aligned} &\frac{1}{a_2}(1+a_1^2+a_2^2) = 2 + \frac{(1+\rho_1^2-2T)(1+\rho_2^2-2S)}{\rho_1\rho_2} - \frac{U^2}{\rho_1\rho_2} \\ &\text{and} \\ &\frac{a_1}{a_2}(1+a_2) = -\frac{(1+\rho_1^2-2T)}{\rho_1} - \frac{(1+\rho_2^2-2S)}{\rho_2} \quad . \end{aligned}\right\} \tag{III.49}$$

Now

$$[(b-2\rho_1)(d-2\rho_2)-U^2]$$

$$= bd + 2\rho_1\rho_2-U^2 - 2(\rho_1 d+\rho_2 b) + 2\rho_1\rho_2 \quad ,$$

so that using (III.28),

$$[(b-2\rho_1)(d-2\rho_2)-U^2]$$

$$= \rho_1\rho_2 \left\{ \frac{(1+\rho_1^2-2T)(1+\rho_2^2-2S)}{\rho_1\rho_2} + 2 - \frac{U^2}{\rho_1\rho_2} \right.$$

$$\left. -2\left[\frac{(1+\rho_2^2-2S)}{\rho_2} + \frac{(1+\rho_1^2-2T)}{\rho_1}\right] + 2 \right\}$$

and substituting the relations (III.49), we obtain

$$[(b-2\rho_1)(d-2\rho_2)-U^2]$$

$$= \rho_1\rho_2\left[\frac{1}{a_2}(1+a_1^2+a_2^2) + \frac{2a_1}{a_2}(1+a_2) + 2\right]$$

$$= \frac{\rho_1\rho_2}{a_2}\left[1+a_1^2+a_2^2+2a_1+2a_1a_2+2a_2\right]$$

$$= \frac{\rho_1\rho_2}{a_2}(1+a_1+a_2)^2 \quad . \tag{III.50}$$

Finally, substituting (III.23) and (III.50) in (III.48), we have

$$|\underline{B}| \sim \frac{(1-\rho_1)^2(1-\rho_2)^2\, a_2}{\rho_1\rho_2(1+a_1+a_2)^2}$$

$$\times \frac{\rho_1^n\rho_2^n[a_2(\beta_{11}-a_2)-a_2^2(\beta_{12}-a_1)(\beta_{21}-a_1)]^2}{a_2^n\,(1-a_1+a_2)(1+a_1+a_2)(1-a_2)^2}$$

or

$$|\underline{B}| \sim \frac{\rho_1^{n-1}\ \rho_2^{n-1}\ (1-\rho_1)^2(1-\rho_2)^2}{a_2^{n-1}}$$

$$\times \frac{[a_2(\beta_{11}-a_2)-a_2^2(\beta_{12}-a_1)(\beta_{21}-a_1)]^2}{(1-a_1+a_2)(1+a_1+a_2)^3(1-a_2)^2} , \qquad \text{(III.51)}$$

where β_{11}, β_{12} and β_{21} are given by (III.9), a_2 and a_1 by (III.49) and the error is $O(\frac{1}{n})$.

APPENDIX IV

EVALUATION OF AN INTEGRAL ENCOUNTERED IN THE RENORMALISATION OF THE DENSITY FUNCTION OF r

Consider the integral $J(m,k,\alpha)$,

$$J(m,k,\alpha) = \int_0^1 \frac{(1-r^2)^m \qquad \{[(1+\alpha)^2-4\alpha r^2]^{\frac{1}{2}}+1+\alpha\}^{\frac{1}{2}}}{\{[(1+\alpha)^2-4\alpha r^2]^{\frac{1}{2}}+1-\alpha\}^k[(1+\alpha)^2-4\alpha r^2]^{\frac{1}{2}}}\,dr ,$$

where $-1 \leq r \leq 1$, $|\alpha| < 1$, $m < k$.

Let

$$(1 + \alpha)^2 - 4\alpha r^2 = (1 + \alpha\varphi)^2 \quad . \tag{IV.1}$$

Then

$$4\alpha r^2 = (1+\alpha)^2 - (1+\alpha\varphi)^2$$

$$= [1+\alpha-(1+\alpha\varphi)][1+\alpha+(1+\alpha\varphi)]$$

$$= \alpha(1-\varphi)[2+\alpha(1+\varphi)]$$

$$= 2\alpha(1-\varphi)[1+ \tfrac{\alpha}{2}(1+\varphi)] ,$$

$$r^2 = \tfrac{1}{2}(1-\varphi)[1 + \tfrac{\alpha}{2}(1+\varphi)]$$

and

$$r = \frac{1}{\sqrt{2}}(1-\varphi)^{\frac{1}{2}} [1 + \tfrac{\alpha}{2}(1+\varphi)]^{\frac{1}{2}} . \tag{IV.2}$$

Also if

$$(1+\alpha)^2 - 4\alpha r^2 = (1+\alpha\varphi)^2$$

$$1 + 2\alpha + \alpha^2 - 4\alpha r^2 = 1 + 2\alpha\varphi + \alpha^2\varphi^2$$

$$2\alpha - 4\alpha r^2 = \alpha[\alpha\varphi^2 + 2\varphi - \alpha]$$

$$4\alpha - 4\alpha r^2 = \alpha[\alpha\varphi^2 + 2\varphi - \alpha + 2]$$

$$4\alpha(1-r^2) = \alpha[2(1+\varphi) - \alpha(1+\varphi)(1-\varphi)]$$

$$= 2\alpha(1+\varphi)[1 - \tfrac{\alpha}{2}(1-\varphi)] ,$$

then

$$1-r^2 = \frac{1}{2}(1+\varphi)[1 - \frac{\alpha}{2}(1-\varphi)] \qquad (IV.3)$$

$$\{[(1+\alpha)^2 - 4\alpha r^2]^{\frac{1}{2}} + 1 - \alpha\} = (1+\alpha\varphi) + 1 - \alpha$$

$$= 2[1 - \tfrac{\alpha}{2}(1-\varphi)] , \qquad (IV.4)$$

$$\{[(1+\alpha)^2 - 4\alpha r^2]^{\frac{1}{2}} + 1 + \alpha\} = (1+\alpha\varphi) + 1 + \alpha$$

$$= 2[1 + \tfrac{\alpha}{2}(1+\varphi)] \qquad (IV.5)$$

and

$$[(1+\alpha)^2 - 4\alpha r^2]^{\frac{1}{2}} = (1+\alpha\varphi) . \qquad (IV.6)$$

From (IV.1) and (IV.2), we have

$$(1+\alpha)^2 - 4\alpha r^2 = (1+\alpha\varphi)^2 ,$$

$$-8\alpha r dr = 2(1+\alpha\varphi)\alpha \, d\varphi ,$$

$$dr = -\frac{1+\alpha\varphi}{4r} d\varphi$$

and so

$$dr = \frac{-(1+\alpha\varphi)}{2^{3/2}(1-\varphi)^{\frac{1}{2}}[1+\frac{\alpha}{2}(1+\varphi)]^{\frac{1}{2}}} d\varphi . \qquad (IV.7)$$

$$\left.\begin{array}{lll} \text{Also at} \quad r = 0, & (1+\alpha)^2 = (1+\alpha\varphi)^2 & \text{so} \quad \varphi = 1 , \\ \text{and at} \quad r = 1, & (1+\alpha)^2 - 4\alpha = (1+\alpha\varphi)^2 , & \\ & (1-\alpha)^2 = (1+\alpha\varphi)^2 & \text{so} \ \varphi = -1 . \end{array}\right\} \qquad (IV.8)$$

Now substituting (IV.3), (IV.4), (IV.5), (IV.6), (IV.7) and (IV.8) in $J(m,k,\alpha)$ we get

$$J(m,k,\alpha) = \int_{-1}^{1} \frac{(1+\varphi)^m [1-\frac{\alpha}{2}(1-\varphi)]^m 2^{\frac{1}{2}}[1+\frac{\alpha}{2}(1+\varphi)]^{\frac{1}{2}}(1+\alpha\varphi)}{2^m\, 2^k[1-\frac{\alpha}{2}(1-\varphi)]^k\,(1+\alpha\varphi)\, 2^{3/2}(1-\varphi)^{\frac{1}{2}}[1+\frac{\alpha}{2}(1+\varphi)]^{\frac{1}{2}}}\, d\varphi$$

$$= \frac{1}{2^{m+k+1}} \int_{-1}^{1} \frac{(1+\varphi)^m}{[1-\frac{\alpha}{2}(1-\varphi)]^{k-m}\,(1-\varphi)^{\frac{1}{2}}}\, d\varphi \,. \qquad \text{(IV.9)}$$

Let $\varphi = 2\psi - 1$, then $\psi = \frac{1}{2}(1+\varphi)$, $d\varphi = 2d\psi$

$1 - \varphi = 2(1-\psi)$, $1 + \varphi = 2\psi$

and at $\varphi = -1$, $\psi = 0$

at $\varphi = 1$, $\psi = 1$.

With this substitution (IV.9) becomes

$$J(m,k,\alpha) = \frac{1}{2^{m+k+1}} \int_{0}^{1} \frac{2^m \psi^m}{[1-\alpha(1-\psi)]^{k-m}}\, \frac{2}{2^{\frac{1}{2}}(1-\psi)^{\frac{1}{2}}}\, d\psi$$

$$= \frac{1}{2^{k+\frac{1}{2}}} \int_{0}^{1} \frac{\psi^m}{[1-\alpha(1-\psi)]^{k-m}(1-\psi)^{\frac{1}{2}}}\, d\psi \,. \qquad \text{(IV.10)}$$

Let $\zeta = 1 - \psi$, $d\zeta = -d\psi$

then at $\psi = 0$, $\zeta = 1$

and at $\psi = 1$, $\zeta = 0$.

Thus (IV.10) becomes

$$J(m,k,\alpha) = \frac{1}{2^{k+\frac{1}{2}}} \int_{0}^{1} (1-\zeta)^m\, \zeta^{-\frac{1}{2}}\, (1-\zeta)^{-(k-m)}\, d\zeta \,.$$

By the theory of the hypergeometric function (see, Rainville [16] Thm. 16)

$$F(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\,\Gamma(c-b)} \int_{0}^{1} t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\, dt \,,$$

if $|z| < 1$ and $Re(c) > Re(b) > 0$.

Thus

$$J(m,k,\alpha) = \frac{1}{2^{k+\frac{1}{2}}} \cdot \frac{\Gamma(\frac{1}{2})\Gamma(m+1)}{\Gamma(m+\frac{3}{2})} F[k-m, \frac{1}{2} ; m+\frac{3}{2} ; \alpha]$$

$$= 2^{-(k+\frac{1}{2})} B(m+1,\frac{1}{2}) F(k-m,\frac{1}{2}; m+\frac{3}{2} ; \alpha) \quad . \tag{IV.11}$$

Consider the case $k = 2m + \frac{3}{2}$. Then (IV.11) is of form

$$J(m,2m+\frac{3}{2}, \alpha) = 2^{-(2m+2)} B(m+1,\frac{1}{2}) F(m+\frac{3}{2}, \frac{1}{2} ; m+\frac{3}{2} ; \alpha) \quad . \tag{IV.12}$$

By a well known result [Rainville [16], Thm. 21]

$$F(a,b;c,z) = (1-z)^{c-a-b} F(c-a,c-b;c;z) ,$$

we see that

$$F(m+\frac{3}{2}, \frac{1}{2}; m+\frac{3}{2}; \alpha) = (1-\alpha)^{-\frac{1}{2}} F(0, m+\frac{1}{2}; m+\frac{3}{2}; \alpha)$$

$$= (1-\alpha)^{-\frac{1}{2}} \ .$$

Thus

$$J(m,2m+\frac{3}{2}, \alpha) = 2^{-(2m+2)} B(m+1, \frac{1}{2})(1-\alpha)^{-\frac{1}{2}} \ . \tag{IV.13}$$

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